

Applied Nonlinear Dynamics

Lecture Notes

Dr. Alexander Schaum

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Dr. Alexander Schaum

Lehrstuhl für Regelungstechnik

Christian-Albrechts-Universität zu Kiel

Institut für Elektrotechnik und Informationstechnik

Technische Fakultät

Kaiserstraße 2

D-24143 Kiel

✉ alsc@tf.uni-kiel.de

🌐 <http://www.control.tf.uni-kiel.de>

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Preface

Nonlinear dynamics has turned out to be an important analysis and design tool for engineering and science applications [Str94], with application areas ranging from chemical reactor engineering [Ari69], [AAS91], biochemical engineering [ZH01; Hen06; SAL12a; SAL12b], mechanical engineering [GKM94], electrical engineering [Kya+13], biology [Mur01], [KS09], physics [WCS98], [BBV08], or social sciences [MNP04]. Understanding the fundamental mechanisms which cause complex nonlinear behavior in dynamical system is a crucial aspect in modern system sciences. Nonlinear phenomena occur in almost any system where complex interactions between different dynamic parts are present. The most fundamental ones are multiplicity, appearance and disappearance of equilibrium solutions, with possibly several simultaneous attractors [Str94]. Such a behavior may introduce severe security risks for some engineering applications like chemical reactors [Tronci], and explain the complexity of predicting the behavior of ecological, meteorological or social systems. Sustained oscillations occurring after some small parameter variation and hysteresis-like phenomena are other typical examples found in many applications.

The present course notes give a short introduction into the basic themes of nonlinear dynamics, following an intuitive, qualitative but sufficiently formal and quantitative approach to the underlying mechanisms typically found in nonlinear systems. This is much in line with an application-oriented thinking, where engineers or scientists are confronted in the daily work with nonlinear phenomena and have to understand why they occur in order to design methods for preventing them, attenuating them, or even introducing them in a controlled fashion.

The aim of the present notes is to provide the basic notions together with a broad catalogue of the most common phenomena, seeking for comprehension of the qualitative-quantitative behavior of solutions of differential equations. Particular emphasis is made to distinguish between local and non-local phenomena, i.e. such involving only the neighborhood of an equilibrium and those occurring within a predefined subset of the state space. Examples of the first kind of phenomenon are the change of stability of an equilibrium point, while the appearance, and disappearance of periodic orbits are examples of non-local phenomena, which often can not be observed if one does not have a non-local picture.

Different case studies are included to illustrate how the theory is applied and discuss its importance. These examples, and the related exercises, should help the student to get a feeling about how the different conceptual results have to be combined in application. A particular attempt was made on discussing the importance of nonlinear dynamics in some aspects of nonlinear control applications.

The manuscript is written for engineers, applied mathematicians, students of physics and related subjects. It is the result of courses held on this subject at the Universidad Autonoma Metropolitana (UAM) - Cuajimalpa in Mexico City in the master programme in natural sciences and engineering (PCNI) from 2013-2014 as well as in the Master of Science program of the Christian-Albrechts-University (CAU) Kiel in the areas of Electrical and information engineering (and business management) in 2014-2015. During these courses very fruitful discussions took place with the students and the scientific staff. In particular I would like to mention at this place Prof. Dr. Jesus Alvarez from the UAM - Iztapalapa, Prof.

Dr. Roberto Bernal from the UAM - Cuajimalpa and Prof. Dr.-Ing. habil. Thomas Meurer from the Chair of Control Engineering at the CAU Kiel.

There are several areas of importance which are not covered in this text. In particular the subjects of weakly nonlinear oscillations, averaging, and a more detailed discussion of chaotic motion and their importance in control engineering.

Given that these lectures notes are still in a preliminary version, comments are welcome at alsc@tf.uni-kiel.de.

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Part I

Continuous-time systems

Basics on Continuous-time Dynamical Systems

In the sequel we will be interested in systems modeled by differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{p}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (2.1)$$

where $\mathbf{x}(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is a real vector-valued function, called the state of the system, $\mathbf{f} \in \mathcal{C}(\mathbb{R}^n \rightarrow \mathbb{R}^n)$ a continuous (nonlinear) function of its arguments defining a vector field, $\mathbf{p}$ is an r -dimensional vector of real-valued constants. The variable t represents time, and t_0 the initial time. Note that in the general time-varying case, the solution will explicitly depend on the initial time. Most applications correspond to time-varying systems, simply, because almost all systems depend on exogenous parameters which typically vary over time. Nevertheless, to understand time-varying systems in first place one has to understand time-invariant systems. In many practical applications, the behavior of the time-varying case represents some kind of perturbation of the time-invariant one, and thus can be delimited by the properties of the later one. For this reason, in most of the present study, we will restrict our attention to the case of time-invariant systems, i.e. where $\mathbf{f}$ does not depend explicitly on the time t . In this case (2.1) is simply represented by the dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}), \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (2.2)$$

In many applications the parameter vector $\mathbf{p}$ may be separated into a fixed component $\mathbf{p}_0$, and one which is amenable to manipulation $\mathbf{u}$ and thus can be used for regulation and control purposes, giving rise to the control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}_0, \mathbf{u}), \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (2.3)$$

The following notions, definition and results are quite standard in the theory of differential equations (see e.g. [Har64; Tes12; Per01; Sas99; Kha96; Str94]).

2.1 The flow of a system

A solution of (2.2) will be called a flow $\phi_t(\mathbf{x}_0)$ of the nonlinear system. The flow represents the time evolution of the state vector $\mathbf{x}(t)$ starting at $\mathbf{x}_0$ with parameter $\mathbf{p}$, such that

$$\mathbf{x}(t; \mathbf{x}_0, \mathbf{p}) = \phi_t(\mathbf{x}_0, \mathbf{p}). \quad (2.4)$$

The meaning of flow is quite intuitive, thinking about a leaf which falls into a river at time $t = 0$ at position $\mathbf{x}_0$, and being driven along flow lines (mathematically parameterized by the function $\phi_t(\mathbf{x}_0)$), such that the actual position of the leaf at time $t \geq 0$ is given exactly by the relation (2.4). It is quite natural that the flow should satisfy the following properties, called flow axioms (see Figure 2.1, and compare with the ones presented in Section 3.1 for linear time-invariant systems):

$$(I) \phi_0(x_0, p) = x_0, \quad (II) \phi_{t_1}[\phi_{t_2}(x_0, p), p] = \phi_{t_1+t_2}(x_0, p) \quad (2.5)$$

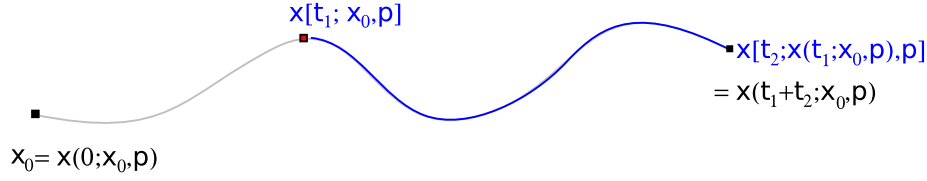


Figure 2.1: Illustration of the flow axioms.

The reason to consider the flow, is that it allows to talk about the properties of the solution of the differential equation (2.2), rather than about the equation itself, i.e. it is more oriented toward the description of phenomena, rather than their causes (described by f in (2.2)).

In virtue of (2.4), the flow is a solution of the differential equation (2.2), i.e.

$$\frac{d\phi_t(x_0)}{dt} = f[\phi_t(x_0), p]. \quad (2.6)$$

Note that in the time-variant case (2.1) the concept of flow is the same, but the function ϕ depends explicitly on the initial time t_0 , i.e. $x(t) = \phi_t(t_0, x_0)$.

Regarding the differential equation (2.2), it is clear by integration that the solution $x(t)$ has to satisfy the integral equation

$$x(t) = x_0 + \int_0^t f[\tau, x(\tau), p] d\tau. \quad (2.7)$$

This representation is sometimes useful to understand some fundamental properties of flows. For example it already establishes that the function f has to be piecewise continuous over time. In general, it is not simpler to solve the integral equation (2.7), than the differential equation (2.1), but the integral form (2.7) is useful to understand some basic facts. For example, in the scalar case with $n = 1$, the scalar function $x(t)$ will increment over time as long as the associated right-hand side f is positive. For the case $n = 1$ this is

$$x(t + dt) \geq x(t) \Leftrightarrow f(x(t), p) \geq 0, \quad dt \geq 0. \quad (2.8)$$

This means that qualitative properties of the flow of the system can be already established by only analyzing the right-hand side f of the dynamics (2.2).

In the general, n -dimensional case, the right-hand side defines a vector field over the state space $\mathbb{R}^n$ (or a compact subset of practical interest), i.e. for any point x in state space, it associates a vector $f(x, p)$ which only depends on the parameter p (or in the time-varying case also on t). In terms of the vector field f , the dynamics equations (2.2) (or (2.1) for the time varying case) and in particular the equation for the time-derivative of the flow in (2.6) mean that the solution curve in state space, i.e. the graph of ϕ_t , is tangential to the vector $f(x)$ at the point x (at time t). See Figure 2.2 for a graphical illustration in the case $n = 2$. This implies that if one knows the vector field, one already knows the qualitative solution curves (without requiring any analytic solution). This fact will be of paramount importance for the qualitative studies on dynamical systems which will be discussed throughout this text.

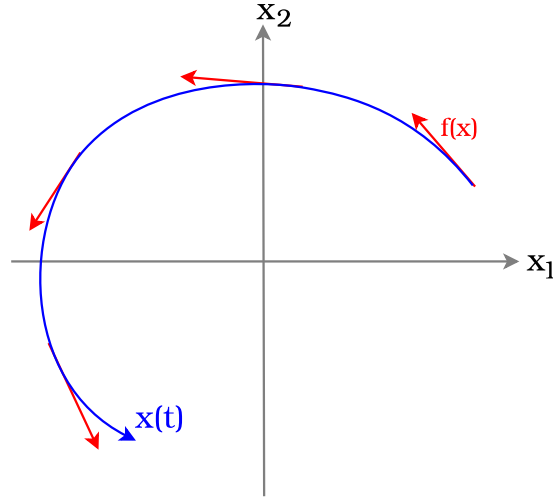


Figure 2.2: Graphical illustration of the vector field and the solution trajectories $x(t)$ for $n = 2$.

2.2 Existence and uniqueness

So far we assumed that the solution of (2.2) existed and was unique. To understand why existence and uniqueness are non-trivial properties which have to be studied carefully, consider the following examples.

2.2.1 Global existence - A counterexample

Consider the nonlinear system with dynamics

$$\dot{x} = x^2, \quad x(0) = x_0. \quad (2.9)$$

From separation of variables and integration one obtains the solution (the reader is encouraged to verify this)

$$x(t) = x_0 \frac{1}{1 - x_0 t}. \quad (2.10)$$

It is clear from the preceding equation that $x(t)$ tends to infinity with t approaching x_0^{-1} , i.e.

$$\lim_{t \rightarrow x_0^{-1}} |x(t)| = \infty$$

as illustrated in Figure 2.3. This property is known as **finite-escape**, and means that the solution does not exist over arbitrary time intervals, but only locally over the open interval $t \in [0, x_0^{-1})$. This phenomenon is typical for nonlinear systems, and has to be accounted for in applications.

2.2.2 Uniqueness - A counterexample

Consider the dynamical system

$$\dot{x} = x^{1/3}, \quad x(0) = 0 \quad (2.11)$$

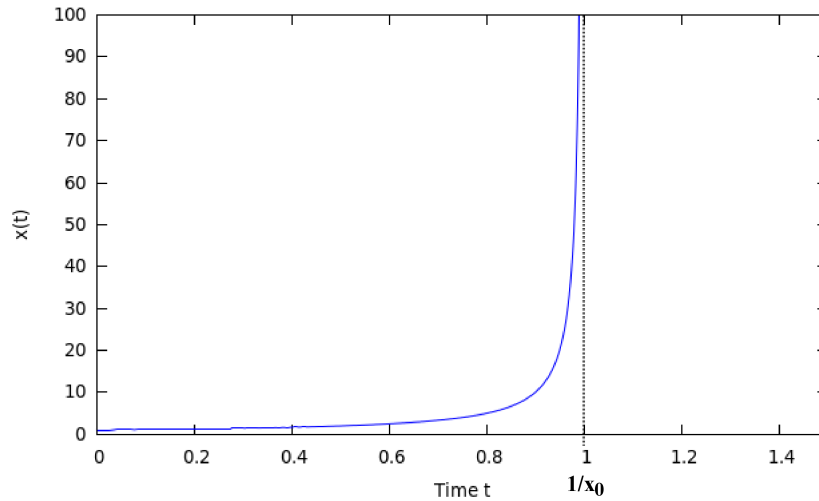


Figure 2.3: Graph of the solution $x(t)$ of equation (2.9), with finite-escape time x_0^{-1} .

starting from the origin $x = 0$ at $t = 0$. Obviously one solution is given by $x(t) = 0 \forall t \geq 0$. Nevertheless, integration using separation of variables yields

$$\int_0^{x(t)} \frac{dx}{x^{1/3}} = \frac{3}{2} x(t)^{2/3} = t, \quad \Leftrightarrow \quad x(t) = \left(\frac{2}{3}t\right)^{3/2},$$

which is different from zero for $t \neq 0$. This means that there are two different solutions. Thus, the question arises which of both trajectories the system will follow, i.e. which of both the flow ϕ_t will correspond to. The two solutions of (2.11) are shown in Figure 2.4. The problem of non-uniqueness is a

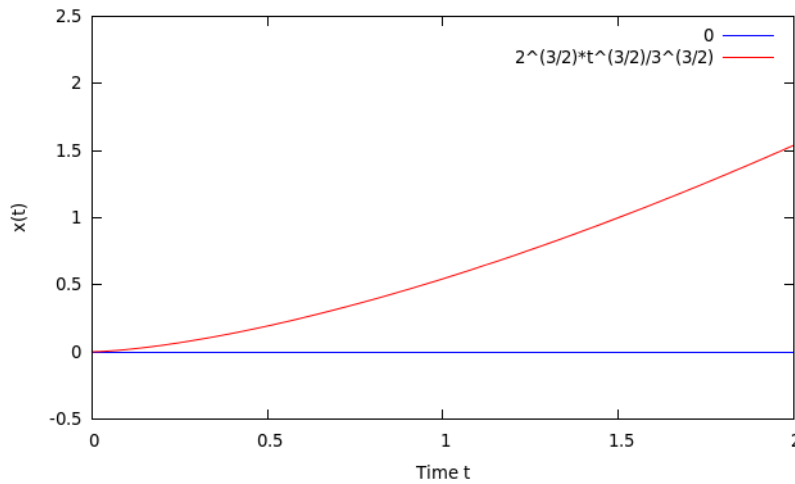


Figure 2.4: Graph of the two solutions of equation (2.11) $x(t) = 0$, and $x(t) = (2/3t)^{3/2}$.

mathematical one, and its implications are probably more about the correct way to model a dynamical system, then on the system itself. Nevertheless, the problem of non-uniqueness of solutions is a serious one which has to be accounted for carefully.

2.2.3 The existence and uniqueness theorem

The following theorem accounts to Picard and Lindelöf, and states sufficient conditions for the existence and uniqueness of local (in time) solutions for general n -dimensional systems of the form

(2.2). Proofs of the theorem can be found in almost any standard text on differential equations (see e.g. [Tes12; Har64; Per01; Str94]).

Theorem 2.1

If f is Lipschitz in x and continuous in t , then there exists a $\tau > 0$ so that the initial value problem (2.2) has a unique solution $x(t) = \phi_t(t_0, x_0)$ over the time interval $t \in [t_0 - \tau, t_0 + \tau]$.

This theorem allows for finite-escape times, actually in example (2.9) the right-hand-side function f is Lipschitz, but it does not allow for multiple solutions as in example (2.11). Analyzing (2.11), one notes that the right-hand-side f is not Lipschitz at $x = 0$, and thus the Picard-Lindelöf theorem does not apply.

2.3 Stability

A fundamental property of linear and nonlinear systems (or flows) is the stability of equilibrium points, i.e. solutions x^* of the algebraic equation

$$0 = f(x^*, p). \quad (2.12)$$

Throughout this text we will be concerned with the stability and need a formal definition. Here, we employ the common definitions associated to A. Lyapunov.

Definition 2.1

An equilibrium point x^* of (2.2) is said to be stable, if for any $\epsilon > 0$ there exists a constant $\delta > 0$ such that for any initial deviation from equilibrium within a δ -neighborhood, the trajectory is comprised within an ϵ -neighborhood, i.e.

$$\forall x_0 : \|x_0 - x^*\| \leq \delta \Rightarrow \|x(t) - x^*\| \leq \epsilon \quad \forall t \geq 0. \quad (2.13)$$

If x^* is not stable, it is referred to as unstable.

This concept is illustrated in Figure 2.5 (left), and is sometimes referred to stability in the sense of Lyapunov. Stability implies that the solutions stay arbitrarily close to the equilibrium whenever the

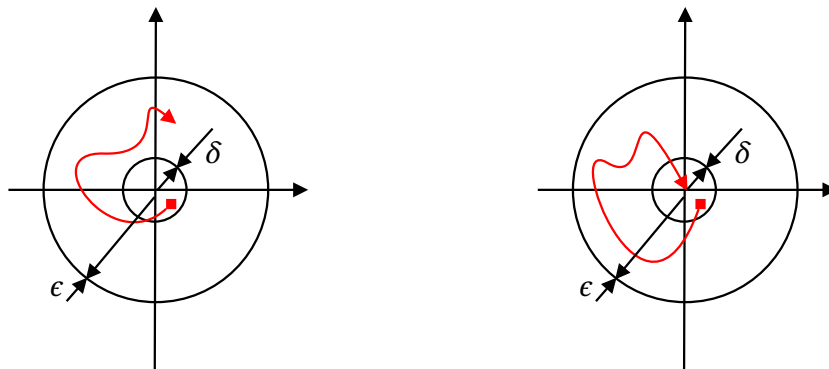


Figure 2.5: Qualitative illustration of the concepts of stability (left) and asymptotic stability (right). initial condition is chosen sufficiently close to the equilibrium point. Note that stability implies

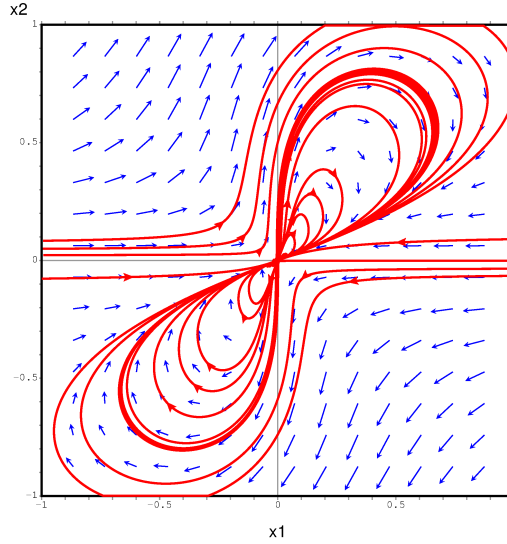


Figure 2.6: Phase portrait of the Vinograd system (2.15), with an unstable but attractive equilibrium point at $\mathbf{x}^* = \mathbf{0}$.

boundedness of solutions, but not that these converge to an equilibrium point. Convergence in turn is ensured by the concept of attractivity.

Definition 2.2

The equilibrium point $\mathbf{x}^*$ is called an attractor for the set $\mathcal{S}$, if

$$\forall \mathbf{x}_0 \in \mathcal{S} : \lim_{t \rightarrow \infty} \|\mathbf{x}(t) - \mathbf{x}^*\| = 0. \quad (2.14)$$

The set $\mathcal{S}$ is called the *domain of attraction*. Note that an equilibrium point may be attractive without being stable, i.e. that trajectories always have a large transient, so that for small ϵ no trajectory will stay for all times within the ϵ -neighborhood, but will return to it and converge to the equilibrium point. An example of such a behavior is given by Vinograd's system [Vin57]

$$\begin{aligned} \dot{x}_1 &= \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)} \\ \dot{x}_2 &= \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)} \end{aligned} \quad (2.15)$$

with phase portrait shown in Figure 2.6, illustrating a butterfly-shaped behavior where, for very small the initial deviation from equilibrium will be, there is a large transient returning asymptotically to the equilibrium point $\mathbf{x}^* = \mathbf{0}$.

It should be mentioned, that if one can demonstrate convergence within a domain $\mathcal{S}_1$, this does not necessarily imply that $\mathcal{S}_1$ is the maximal domain of attraction. The determination of the maximal domain of attraction for an attractive equilibrium of a nonlinear system is, in general, a non-trivial task, due to the fact that, in contrast to linear systems, nonlinear systems can have multiple attractors, each one with its own domain of attraction. This issue will be analyzed later with more detail.

A common, similar concept, including a statement for the transient behavior, is called asymptotic stability and is defined next.

Definition 2.3

An equilibrium point $\mathbf{x}^*$ of (2.2) is said to be asymptotically stable if it is stable and attractive.

This concept is illustrated in Figure 2.5 (right). Note that in contrast to pure attractivity, the concept of asymptotic stability does not allow for large transients associated to small initial deviations, and is thus much stronger, and more of practical interest.

The asymptotic stability does not state anything about convergence speed. It only establishes that after an infinite time period the solution $\mathbf{x}(t)$, if starting in the domain of attraction, will approach $\mathbf{x}^*$ without ever reaching it actually. A concept which allows to overcome this issue is the stronger one of exponential stability.

Definition 2.4

An equilibrium point $\mathbf{x}^*$ of (2.2) is said to be exponentially stable in a set $\mathcal{S}$, if it is stable, and there are constants $a, \lambda > 0$ so that

$$\forall \mathbf{x}_0 \in \mathcal{S} : \|\mathbf{x}(t) - \mathbf{x}^*\| \leq a \|\mathbf{x}_0 - \mathbf{x}^*\| e^{-\lambda t}. \quad (2.16)$$

The constant a is known as the amplitude, and λ as the convergence rate.

Clearly, exponential stability implies asymptotic stability. It is quite noteworthy that, unless the convergence is still asymptotic, for exponentially stable equilibria it is possible to determine exactly the time needed to approach the equilibrium up to a given distance. In practice one often considers convergence if a 98.5% approximation is achieved¹. Accordingly the required convergence time can be determined as follows. Suppose that the bound (2.16) holds with $a = 1$ and is exact, i.e. it holds with equality. In this case the notion of characteristic time t_c is useful

$$t_c = \lambda^{-1} \quad (2.17)$$

Note that the equality in (2.16) implies that the norm behaves like a linear system, as actually

$$\frac{d\|\mathbf{x}(t) - \mathbf{x}^*\|}{dt} \leq -\lambda \|\mathbf{x}(t) - \mathbf{x}^*\|.$$

Note that the characteristic time t_c corresponds to the inverse slope of the time response of this linear system at $t = 0$ (see Figure 2.7). Denote by t_s the settling time, i.e. the time required for 98.5% convergence. It follows that

$$\frac{\|\mathbf{x}(t_s) - \mathbf{x}^*\|}{\|\mathbf{x}_0 - \mathbf{x}^*\|} \leq e^{-\lambda t_s} = 0.015 \quad \Leftrightarrow \quad t_s \approx 4 \frac{1}{\lambda} = 4t_c. \quad (2.18)$$

This means that practical convergence is obtained after approximately four characteristic times t_c . This notion is particularly useful in prediction and control applications, when ensured convergence time measures are required. Figure 2.7 illustrates the concept of characteristic (t_c) and settling (t_s) times.

¹To understand this fact note that two curves which are identical up to 98.5% are hardly distinguishable with the naked eye. Furthermore, in practice there is always some *noisy* variation in the system, causing fluctuations of the trajectories.

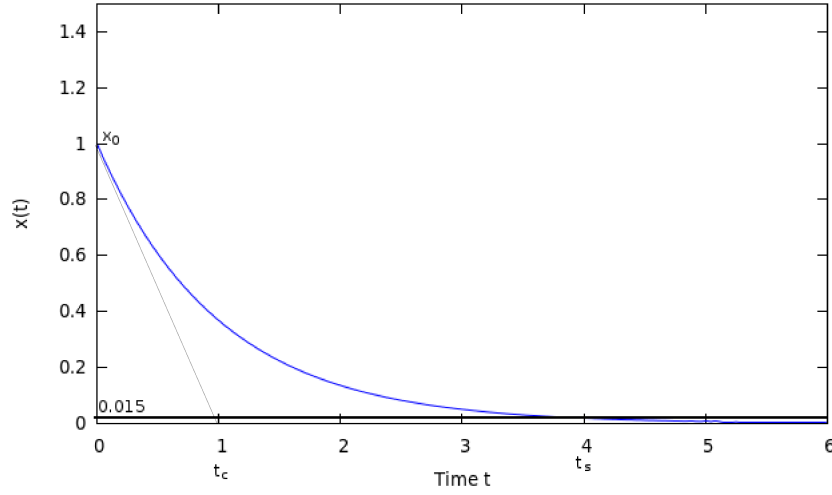


Figure 2.7: Illustration of the concepts of characteristic (t_c) and settling (t_s) times for a linear system.

Some words are in order on the amplitude constant a in (2.16). Clearly, $a \geq 1$. Note that for any $a \geq 1$ oscillations are possible (see Figure 2.8). While for $a = 1$, the convergence of the norm of $\mathbf{x}$ is monotonically decreasing, for $a > 1$ an initial overshoot is possible. This kind of behavior is typical in applications, and corresponds to the one qualitatively depicted for the asymptotic stability case (right side) in Figure 2.5.

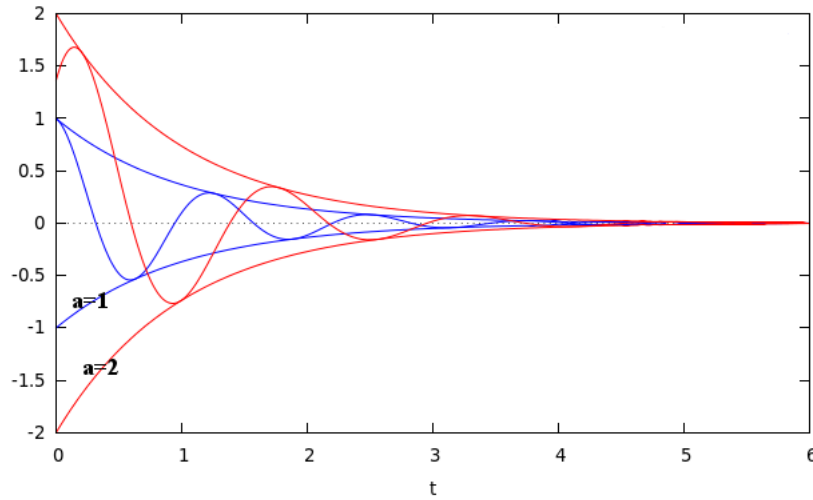


Figure 2.8: Illustration of different exponentially stable time responses for $a = 1$ (blue) and $a = 2$ (red, with characteristic time $t_c = 1$)

To conclude this section, note that all the above stability and attractivity concepts can also be applied to sets. To exemplify this, and for later reference, consider the definition of an attractive compact set.

Definition 2.5

A compact set $\mathcal{M}$ is called attractive for the domain $\mathcal{S}$, if

$$\forall \mathbf{x}_0 \in \mathcal{S} : \lim_{t \rightarrow \infty} \mathbf{x}(t) \in \mathcal{M}.$$

Convergence to a set is of practical interest because in many situations it may be used to obtain a reduced model for analysis and design purposes, or may even be part of the design as in the *sliding-mode control* approach (see e.g. [Utk92]). In many situations it is an essential part of the stability assessment of a dynamical system [Sei69]. A simple example is provided by the linear system

$$\dot{x}_1 = -x_1 + x_2, \quad x_1(0) = x_{10} \quad (2.19)$$

$$\dot{x}_2 = 0, \quad x_2(0) = x_{20} \quad (2.20)$$

with solution (the reader should verify this)

$$x_1(t) = e^{-t}(x_{10} - x_{20}), \quad x_2(t) = x_{20}.$$

This means that x_2 remains constant over time, and x_1 just converges towards the initial value of x_2 . The associated set of equilibrium points is the continuous straight diagonal line set in the (x_1, x_2) -plane defined by

$$\mathcal{M} = \{x \in \mathbb{R}^2 \mid x_1 = x_2\} \quad (2.21)$$

and for any initial condition pair (x_{10}, x_{20}) the solution converges asymptotically towards this set as qualitatively depicted in Figure 2.9.

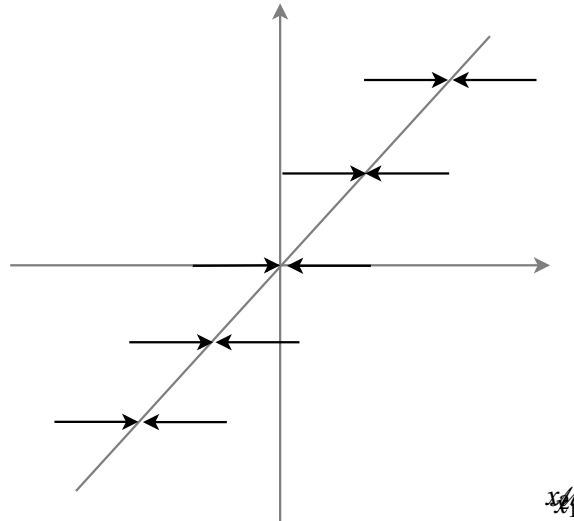


Figure 2.9: Qualitative sketch of the solution trajectories $x(t)$ of the dynamical system (2.19), converging to the diagonal set $\mathcal{M}$.

Furthermore, in many applications it is more important to show the convergence to a set rather than to an equilibrium point. This is due to the fact that in application situations the exact system parameters are often not known exactly, but within some tolerance interval, defining on the other hand a tolerance interval for the required convergence. This requirement is conceptually defined via the practical stability notion introduced by Lefschetz [SL61]. Consider a nominal parameter $\bar{p}$ for which the operation point $\bar{x}$ of (2.2) is defined. Then consider a (possibly time varying) deviated parameter $p(t)$ with which the system is actually operating. Then practical stability is defined as follows.

Definition 2.6

The operation point $\bar{x}$ is said to be practically stable if for given parameter and initial condition deviation sizes δ_p and δ_0 , respectively, it holds that

$$\forall t \geq 0: \|\mathbf{p}(t) - \bar{\mathbf{p}}\| \leq \delta_p, \|\mathbf{x}_0 - \bar{\mathbf{x}}\| \leq \delta_x \Rightarrow \|\mathbf{x}(t) - \bar{\mathbf{x}}\| \leq \alpha(\|\mathbf{x}_0 - \bar{\mathbf{x}}\|, t) + \beta(\delta_p), \quad (2.22)$$

with α being a increasing-decreasing (i.e. class $\mathcal{KL}$) and β an increasing (i.e. class $\mathcal{K}$) function^a.

^aClass $\mathcal{KL}$ are monotonically increasing in the first and decreasing in the second argument, while class $\mathcal{K}$ functions are monotonically increasing

Geometrically speaking, practical stability ensures the boundedness of solutions in presence of bounded perturbations, and that the initial condition deviation asymptotically vanishes. In fact, for $\delta_p = 0$ the concept of asymptotic stability (see Definition 2.3) is recovered.

Finally, it will be helpful in the sequel to understand the concept of (positively) invariant sets $\mathcal{M}$ for a dynamical system.

Definition 2.7

A set $\mathcal{M} \subset \mathbb{R}^n$ is called positively invariant, if for all $\mathbf{x}_0 \in \mathcal{M}$ it holds that $\mathbf{x}(t; \mathbf{x}_0) \in \mathcal{M}$ for all $t \geq 0$.

The distinction positive invariant is used to distinguish the concept from negative invariance, referring to a reversion of time (i.e., letting time tending to minus infinity). As in this course we are interested normally in the positive sense of time, positive invariance will be denoted just by invariance. The reader should convince himself that the set $\mathcal{M}$ defined in (2.21) is an invariant set for the dynamics (2.19) (see Figure 2.9).

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Linear systems revisited

In this chapter, basic results from linear systems theory are provided, starting with the general solution of autonomous and non-autonomous linear odes, and finishing with linear systems with time-varying coefficients.

3.1 Linear Time-Invariant (LTI) systems

A dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x} \in \mathbb{R}^n, \mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (3.1)$$

is called linear if $\mathbf{f}$ is a linear function. A particular case for a linear system is

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (3.2)$$

and $A \in \mathbb{R}^{n \times n}$ is a linear map (or transformation). Formally, the solution can be written as

$$\mathbf{x}(t) = \exp(At)\mathbf{x}_0 \quad (3.3)$$

giving rise to the definition of the matrix exponential (the so-called *fundamental solution*)

$$S(t) = \exp(At) = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}, \quad \mathbf{x}(t) = S(t)\mathbf{x}_0. \quad (3.4)$$

It follows that (the reader should convince himself that this property follows from the definition of $S(t)$)

$$\frac{dS(t)}{dt} = AS(t) \quad (3.5)$$

so that

$$\frac{d\mathbf{x}(t)}{dt} = \frac{dS(t)}{dt}\mathbf{x}_0 = AS(t)\mathbf{x}_0 = A\mathbf{x}(t),$$

showing that (3.3) is effectively a solution of (3.2). It can be further shown that this solution is unique, but for the purpose at hand it is sufficient to notice that the function $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ is Lipschitz continuous and thus the uniqueness follows from Theorem 2.1.

The characteristic equation of A is given by

$$\det(\lambda I - A) = \sum_{i=0}^n a_i \lambda^i = 0. \quad (3.6)$$

By Cailey-Hamilton's theorem [Fis09] it follows that the matrix A satisfies its own characteristic equation (3.6), i.e.

$$\sum_{i=0}^n a_i A^i = 0.$$

As a consequence there exist constants c_{mi} such that

$$\forall m \geq n : A^m = \sum_{i=0}^{n-1} c_{mi} A^i$$

Accordingly, the series in (3.4) can be expressed as a sum of the first n powers of A , with adequately chosen coefficients

$$S(t) = \exp(At) = \sum_{i=0}^{n-1} \kappa_i(t) A^i, \quad (3.7)$$

showing that the matrix exponential can be computed by a finite number of matrix multiplications and that it attains finite values.

The matrix function $S(t)$, i.e. the fundamental solution of (3.2) satisfies the following properties:

- (i) $S(0) = I$
- (ii) $S(t_1)S(t_0) = S(t_1 + t_0)$
- (iii) $S(t - t_0)^{-1} = S(t_0 - t)$

which are also known as flow axioms (cp. with (2.4)). Accordingly, the fundamental solution $S(t)$ is associated to the flow of the system

$$\phi_t(\mathbf{x}_0) = S(t)\mathbf{x}_0. \quad (3.8)$$

For the case of a non-autonomous (i.e. time-varying) equation with constant matrix A ,

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{u}(t) \quad (3.9)$$

with $\mathbf{u}(t) \in \mathbb{R}^n$, multiply both sides with $\exp(-At)$ to obtain

$$e^{-At} \dot{\mathbf{x}}(t) = e^{-At} A\mathbf{x}(t) + e^{-At} \mathbf{u}(t) = -\left(\frac{d}{dt}(e^{-At})\right) \mathbf{x}(t) + e^{-At} \mathbf{u}(t)$$

or equivalently, recalling the product formula for differentiation

$$\frac{d}{dt}(e^{-At} \mathbf{x}(t)) = e^{-At} \mathbf{u}(t).$$

Integrate from 0 to t

$$\int_0^t \frac{d}{d\tau}(e^{-A\tau} \mathbf{x}(\tau)) d\tau = e^{-A\tau} \mathbf{x}(\tau) \Big|_0^t = e^{-At} \mathbf{x}(t) - \mathbf{x}_0 = \int_0^t e^{-A\tau} \mathbf{u}(\tau) d\tau,$$

rearrange, and multiply with $S(t) = \exp(At)$ to obtain

$$\dot{\mathbf{x}}(t) = \exp(At)\dot{\mathbf{x}}_0 + \int_0^t \exp(A(t-\tau))\mathbf{u}(\tau)d\tau,$$

or equivalently

$$\mathbf{x}(t) = S(t)\mathbf{x}_0 + \int_0^t S(t-\tau)\mathbf{u}(\tau)d\tau, \quad S(t) = e^{At}, \quad (3.10)$$

what is known as the *variation of constants formula*.

Next, consider a state transformation $\xi = T\mathbf{x}$ with T being an invertible matrix (i.e., an element of the orthogonal group $\mathcal{O}(\mathbb{R}^n)$). It follows that

$$\dot{\xi} = T\dot{\mathbf{x}} = T(A\mathbf{x}) = TAT^{-1}\xi, \quad \xi(0) = \xi_0 = T\mathbf{x}_0 \quad (3.11)$$

and

$$\xi(t) = S_\xi(t)\xi_0 = \exp(TAT^{-1}t)\xi_0 \quad (3.12)$$

implying that for any $T \in \mathcal{O}(\mathbb{R}^n)$ we have that

$$\mathbf{x}(t) = T^{-1}\xi(t) = T^{-1}\exp(TAT^{-1}t)T\mathbf{x}_0. \quad (3.13)$$

This fact is particularly useful when thinking about diagonalization and transformations into Jordan normal forms. To see this in more detail, recall (see e.g. [Fis09]) that the eigenvalues $\lambda_i, i = 1, \dots, n$ are the solutions of the characteristic equation (3.6) that can be written in factor form as

$$\det(\lambda I - A) = \prod_{i=1}^n (\lambda - \lambda_i) = 0.$$

The number of times a given eigenvalue appears as solution in the preceding equation is called the *algebraic multiplicity* α_i of this eigenvalue. It always holds that

$$\sum_{k=1}^{n_\lambda} \alpha_k = n,$$

because the characteristic polynomial has exactly n roots and there are n_λ different eigenvalues.

Each eigenvalue λ_i has associated an eigenvector $\mathbf{v}_i$ according to

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

or equivalently

$$(A - \lambda_i I)\mathbf{v}_i = \mathbf{0}.$$

This means that $\mathbf{v}_i$ is contained in the *kernel* of the matrix $(A - \lambda_i I)$. Denote the *kernel* by

$$X_i = \ker(A - \lambda_i I).$$

The set $X_i \subset \mathbb{R}^n$ is called the *eigenspace* associated to the eigenvalue λ_i and is spanned by all eigenvectors associated to λ_i . The dimension of this eigenspace (i.e. the number of eigenvectors) is called the *geometric multiplicity* γ_i of the eigenvalue λ_i . It always holds that $\gamma_i \leq \alpha_i$.

Note that each eigenvector $\mathbf{v}_k$ introduces a positively invariant subspace

$$\Sigma_k = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = c\mathbf{v}_k, c \in \mathbb{R}\}. \quad (3.14)$$

That this space is positively invariant can be easily seen by considering the vector field in Σ_k . It holds that

$$\mathbf{x} = c\mathbf{v}_k \Rightarrow \dot{\mathbf{x}} = A\mathbf{x} = Ac\mathbf{v}_k = \lambda_k c\mathbf{v}_k = \lambda_k \mathbf{x} \in \Sigma_k.$$

Given that for all $\mathbf{x} \in \Sigma_k$ the vector field points parallel to Σ_k , and the solution trajectories follow tangentially the vector field it results that any trajectory starting in Σ_i stays in Σ_i for all $t \geq 0$. Accordingly, if $\lambda_k < 0$, the vector field points in the inverse direction of $\mathbf{x}$ implying that the $\mathbf{x}$ will decrease in modulus $\|\mathbf{x}\|$ and for $\lambda_k > 0$ it points in the same direction implying an amplification of $\|\mathbf{x}\|$.

If both algebraic and geometric multiplicity are the same, i.e. $\alpha_i = \gamma_i$, then there exist $n = \sum_{k=1}^{n_\lambda} \alpha_k = \sum_{k=1}^{n_\lambda} \gamma_k$ linearly independent eigenvectors which form a basis (not necessarily orthonormal) of the state space and thus the matrix A is diagonalizable. If $\gamma_i < \alpha_i$, then there are only $\sum_{k=1}^{n_\lambda} \gamma_k < n$ linearly independent eigenvalues spanning only a subspace of the state space and the matrix is not diagonalizable. This will be exploited in the following.

As stated above, when $\alpha_k = \gamma_k$, $k = 1, \dots, n_\lambda$ there are n linearly independent eigenvectors $\mathbf{v}_i$, $i = 1, \dots, n$ and the matrix

$$T = [\mathbf{v}_1 \quad \dots \quad \mathbf{v}_n]$$

is invertible (because the column vectors are linearly independent). By definition it holds that

$$AT = [A\mathbf{v}_1 \quad \dots \quad A\mathbf{v}_n] = [\lambda_1 \mathbf{v}_1 \quad \dots \quad \lambda_n \mathbf{v}_n] = T\Lambda \quad (3.15)$$

with the diagonal matrix of eigenvalues

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}.$$

Introduce the state transformation

$$\boldsymbol{\xi} = T^{-1}\mathbf{x}$$

and consider the associated dynamics

$$\dot{\boldsymbol{\xi}} = T^{-1}\dot{\mathbf{x}} = T^{-1}A\mathbf{x} = T^{-1}AT\boldsymbol{\xi}, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 = T^{-1}\mathbf{x}_0.$$

By (3.15) it holds that

$$\dot{\boldsymbol{\xi}} = T^{-1}T\Lambda\boldsymbol{\xi} = \Lambda\boldsymbol{\xi}, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0,$$

or equivalently

$$\dot{\xi}_k = \lambda_k \xi_k, \quad \xi_k(0) = \xi_{k0}$$

with solutions

$$\xi_k(t) = \xi_{k0} e^{\lambda_k t}, \quad k = 1, \dots, n. \quad (3.16)$$

In vector notation this can be written as

$$\boldsymbol{\xi}(t) = \exp(\Lambda t) \boldsymbol{\xi}_0, \quad \exp(\Lambda t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix}.$$

Thus, the exact solution for $\mathbf{x}$ is given by

$$\mathbf{x}(t) = T \exp(\Lambda t) T^{-1} \mathbf{x}_0 \quad (3.17)$$

and can also be written as

$$\mathbf{x}(t) = e^{\lambda_* t} N(t) \mathbf{x}_0, \quad N(t) = e^{-\lambda_* t} T \exp(\Lambda t) T^{-1}$$

with

$$\lambda_* = \max_{i=1, \dots, n} \lambda_i$$

being the largest eigenvalue of the matrix A . Accordingly, it follows that¹

$$\|\mathbf{x}(t)\| \leq e^{\lambda_* t} \|N(t)\| \|\mathbf{x}_0\|$$

and $N(t)$ is bounded, i.e. for all $t \geq 0$ it holds that $\|N(t)\| \leq a$, because λ_* is the largest eigenvalue, so that all $e^{(\lambda_k - \lambda_*)t} \leq 1$. This implies that for $\lambda_* < 0$ the origin is exponentially stable with amplitude a and decay rate $\lambda = |\lambda_*|$.

Alternatively, a similar result can be derived in the following way. When $\gamma_i = \alpha_i$ for all eigenvalues λ_i , then there are n linearly independent eigenvectors. These eigenvectors form a basis of $\mathbb{R}^n$ and thus any solution $\mathbf{x}(t)$ can be viewed as a linear combination

$$\mathbf{x}(t) = \sum_{k=1}^n x_k(t) \mathbf{v}_k, \quad x_k(t) = \langle \mathbf{x}(t), \mathbf{v}_k \rangle, \quad (3.18)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product between two vectors. By the above analysis it holds that

$$\dot{\mathbf{x}} = \sum_{k=1}^n \dot{x}_k \mathbf{v}_k = A \sum_{k=1}^n x_k \mathbf{v}_k = \sum_{k=1}^n x_k A \mathbf{v}_k = \sum_{k=1}^n x_k \lambda_k \mathbf{v}_k.$$

In virtue of the linear independence of the eigenvectors it follows that

$$\dot{x}_k = \lambda_k x_k, \quad x_k(0) = \langle \mathbf{x}_0, \mathbf{v}_k \rangle.$$

¹ The norm of a matrix can be defined as $\sup_{\|\mathbf{x}\| \leq 1} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$.

Accordingly, the solution $\mathbf{x}(t)$ can be written as

$$\mathbf{x}(t) = \sum_{k=1}^n x_{k0} e^{\lambda_k t} \mathbf{v}_k. \quad (3.19)$$

Again, taking norms on both sides and using the triangle inequality it follows that

$$\|\mathbf{x}(t)\| \leq e^{\lambda_* t} \sum_{k=1}^n |x_{k0}| \|\mathbf{v}_k\|.$$

The case $\alpha_k = \gamma_k$ is illustrated for a two-dimensional state space in Figure 3.1 for the cases of two negative eigenvalues (left), and one negative and one positive eigenvalue (right). The first case is known as a (stable) node, and the second one as a saddle².

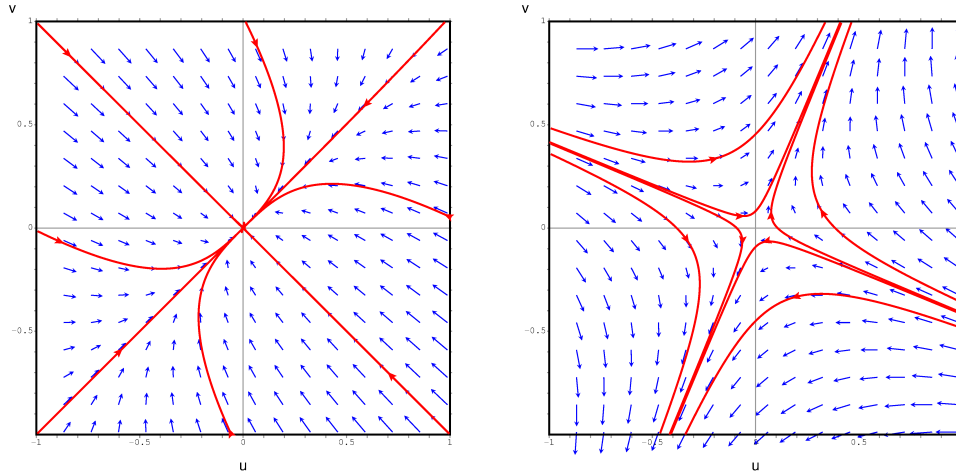


Figure 3.1: Illustration of the behavior with two different eigenvalues: a stable node (left, $\lambda_1 = -1, \lambda_2 = -3$), and a saddle (right, $\lambda_1 = +\sqrt{2}, \lambda_2 = -\sqrt{2}$).

If the algebraic multiplicity α_i is larger than the geometric one γ_i , then the concept of generalized eigenvector is useful to transform the system into the simple structure of the Jordan normal form as illustrated next for the case of an eigenvalue with algebraic multiplicity α_i and geometric multiplicity $\gamma_i = 1$. Then there is only one solution $\mathbf{v}_i$ for the eigenvector equation

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i,$$

or equivalently

$$(A - \lambda_i I)\mathbf{v}_i = \mathbf{0}.$$

A generalized eigenvector of degree r for the eigenvalue λ_i is a vector $\mathbf{h}_{i,r}$ which satisfies

$$(A - \lambda_i I)^k \mathbf{h}_{i,r} \neq \mathbf{0}, \quad k = 1, \dots, r-1, \quad (A - \lambda_i I)^r \mathbf{h}_{i,r} = \mathbf{0}.$$

²This association stems from a comparison of the geometric shape in a three-dimensional set-up, being similar to a horse saddle.

Using this definition, the eigenvector $\mathbf{v}_i$ is a generalized eigenvalue of degree 1. For the determination of the generalized eigenvectors the following sequence is used (where the eigenvector $\mathbf{v}_i$ is denoted by $\mathbf{h}_{i,1}$)

$$\begin{aligned} (A - \lambda_i I)\mathbf{h}_{i,1} &= \mathbf{0}, \\ (A - \lambda_i I)\mathbf{h}_{i,k} &= \mathbf{h}_{i,k-1}, \quad k = 2, \dots, \alpha_i. \end{aligned} \quad (3.20)$$

This sequence is also called a *Jordan chain* or a *chain of generalized eigenvectors*, and can be rewritten as follows

$$A\mathbf{h}_{i,1} = \lambda_i \mathbf{h}_{i,1}, \quad A\mathbf{h}_{i,k} = \lambda_i \mathbf{h}_{i,k} + \mathbf{h}_{i,k-1}, \quad k = 2, \dots, \alpha_i,$$

so that we have

$$A \begin{bmatrix} \mathbf{h}_{i,1} & \mathbf{h}_{i,2} & \cdots & \mathbf{h}_{i,\alpha_i} \end{bmatrix} = \lambda_i \begin{bmatrix} \mathbf{h}_{i,1} & \mathbf{h}_{i,2} & \cdots & \mathbf{h}_{i,\alpha_i} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{h}_{i,1} & \cdots & \mathbf{h}_{i,\alpha_i-1} \end{bmatrix}.$$

This motivates the transformation

$$T = \begin{bmatrix} \mathbf{h}_{i,1} & \mathbf{h}_{i,2} & \cdots & \mathbf{h}_{i,\alpha_i} \end{bmatrix} \quad (3.21)$$

so that

$$AT = T(\lambda_i I + N_{\alpha_i}) \quad (3.22)$$

with

$$N_{\alpha_i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & & \cdots & 0 & 1 \\ 0 & \cdots & & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{\alpha_i \times \alpha_i}.$$

Note that N_{α_i} is nil-potent of order α_i , i.e. $N_{\alpha_i}^{\alpha_i} = 0$. Using the transformation matrix T given in (3.21), the following Jordan block is obtained

$$J_k = T^{-1}AT \stackrel{(3.22)}{=} T^{-1}T(\lambda_i I + N_{\alpha_i}) = \lambda_i I + N_{\alpha_i} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{\alpha_i \times \alpha_i}. \quad (3.23)$$

For a Jordan block of dimension α_i , the associated solution for $\xi_j, j \in \{1, \dots, \alpha_i\}$ then reads (the reader is encouraged to demonstrate this combining the variation of constants formula (3.10) with *complete induction*)

$$\begin{aligned} \xi_j(t) &= e^{\lambda_i t} \left(\xi_{j,0} + \xi_{j+1,0}t + \cdots + \xi_{j+k,0} \frac{t^k}{k!} + \cdots + \xi_{\alpha_i,0} \frac{t^{\alpha_i-j}}{(\alpha_i-j)!} \right) \\ &= e^{\lambda_i t} \sum_{k=0}^{\alpha_i-j} \xi_{j+k,0} \frac{t^k}{k!} \end{aligned} \quad (3.24)$$

so that the matrix $\exp(TAT^{-1})$ can be easily calculated based on the Jordan blocks (3.23). For the particular case $\alpha = n$, just one big Jordan block exists and the solution can be written as

$$\xi(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} & \dots & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \dots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & e^{\lambda t} \end{bmatrix} \xi_0 = \exp(J_\alpha t) \xi_0.$$

The corresponding solution for $\mathbf{x}$ can again be found by the back-transformation

$$\mathbf{x}(t) = T\xi(t) = T \exp(J_\alpha t) T^{-1} \mathbf{x}_0$$

and can be written accordingly as

$$\mathbf{x}(t) = e^{\lambda t} N(t) \mathbf{x}_0, \quad N(t) = e^{-\lambda t} T \exp(J_\alpha t) T^{-1}$$

with a bounded matrix $N(t)$. Thus, the exponential stability follows for $\lambda < 0$.

A typical case for such a dynamics is illustrated in Figure 3.2.

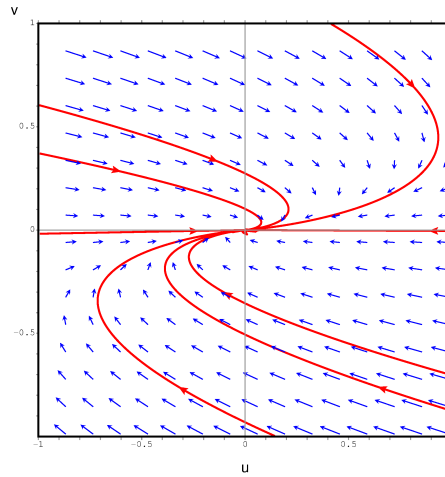


Figure 3.2: Illustration of the behavior associated to a generalized eigenvector with multiplicity two, and eigenvector $\lambda = -1$ (the eigendirection is the x-axis).

If neither of the preceding cases applies, then there are pairs of conjugate complex eigenvalues³ $(\lambda_k, \lambda_{k+1})$ with

$$\lambda_k = \alpha_k + i\beta_k, \quad \lambda_{k+1} = \bar{\lambda}_k = \alpha_k - i\beta_k.$$

Consider the case $n = 2$ and let A have a pair of conjugate complex eigenvalues λ_1, λ_2 with $\lambda_1 = \alpha + i\beta = \bar{\lambda}_2$. Denote the associated conjugate complex eigenvectors as

$$\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}, \quad \mathbf{v}_2 = \mathbf{a} - i\mathbf{b}.$$

The associated eigenvalue problem can be written as

$$A(\mathbf{a} + i\mathbf{b}) = (\alpha + i\beta)(\mathbf{a} + i\mathbf{b}) = \alpha\mathbf{a} - \beta\mathbf{b} + i(\beta\mathbf{a} + \alpha\mathbf{b}) \quad (3.25a)$$

³Note that here it is assumed that the associated complex-conjugated eigenvalues have successive indexes.

$$A(\mathbf{a} - i\mathbf{b}) = (\alpha - i\beta)(\mathbf{a} - i\mathbf{b}) = \alpha\mathbf{a} - \beta\mathbf{b} - i(\beta\mathbf{a} + \alpha\mathbf{b}) \quad (3.25b)$$

Adding (3.25a) and (3.25b) yields

$$2A\mathbf{a} = 2(\alpha\mathbf{a} - \beta\mathbf{b})$$

and subtracting (3.25b) from (3.25a)

$$2iA\mathbf{b} = 2i(\beta\mathbf{a} + \alpha\mathbf{b}).$$

Summarizing these two equations yields

$$A\mathbf{a} = \alpha\mathbf{a} - \beta\mathbf{b}$$

$$A\mathbf{b} = \beta\mathbf{a} + \alpha\mathbf{b},$$

or equivalently (the reader should convince himself of this relation by carrying out the vector-matrix product on the right hand side)

$$A[\mathbf{a} \ \mathbf{b}] = [\mathbf{a} \ \mathbf{b}] \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Thus, introducing the state transformation

$$T = [\mathbf{a} \ \mathbf{b}], \quad \boldsymbol{\xi} = T^{-1}\mathbf{x}$$

it follows that

$$\dot{\boldsymbol{\xi}} = T^{-1}\dot{\mathbf{x}} = T^{-1}A\mathbf{x} = T^{-1}AT\boldsymbol{\xi} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \boldsymbol{\xi}, \quad \boldsymbol{\xi}(0) = T^{-1}\mathbf{x}_0.$$

This is called *real(-valued) Jordan normal form* and can be rewritten as

$$\dot{\xi}_1 = \alpha\xi_1 + \beta\xi_2, \quad \xi_1(0) = \xi_{10} \quad (3.26a)$$

$$\dot{\xi}_2 = -\beta\xi_1 + \alpha\xi_2, \quad \xi_2(0) = \xi_{20}. \quad (3.26b)$$

By the change into polar coordinates

$$r = (\xi_1^2 + \xi_2^2)^{\frac{1}{2}}, \quad \theta = \arctan\left(\frac{\xi_2}{\xi_1}\right) \quad (3.27)$$

these dynamics are equivalent to⁴

$$\dot{r} = \alpha r, \quad r(0) = r_0 \quad (3.28)$$

$$\dot{\theta} = -\beta, \quad \theta(0) = \theta_0 \quad (3.29)$$

with

$$r_0 = \sqrt{\xi_{1,0}^2 + \xi_{2,0}^2}, \quad \theta_0 = \arctan\left(\frac{\xi_{2,0}}{\xi_{1,0}}\right)$$

⁴These equations are easily derived using the following relations (see exercise 1)

$$r\dot{r} = \xi_1\dot{\xi}_1 + \xi_2\dot{\xi}_2, \quad \dot{\theta} = \frac{\xi_1\dot{\xi}_2 - \xi_2\dot{\xi}_1}{r^2}.$$

and solutions

$$r(t) = e^{\alpha t} r_0, \quad \theta(t) = \theta_0 - \beta t, \quad (3.30)$$

corresponding to trajectories spinning counterclockwise ($\beta > 0$) or clockwise ($\beta < 0$) around the origin $[\xi_1, \xi_2]^T = \mathbf{0}$, with period $T = 2\pi/b$, and radius which is decreasing ($\alpha < 0$, stable spiral), constant ($\alpha = 0$, center), or increasing ($\alpha > 0$, unstable spiral) as shown in Figure 3.3. In ξ -coordinates these

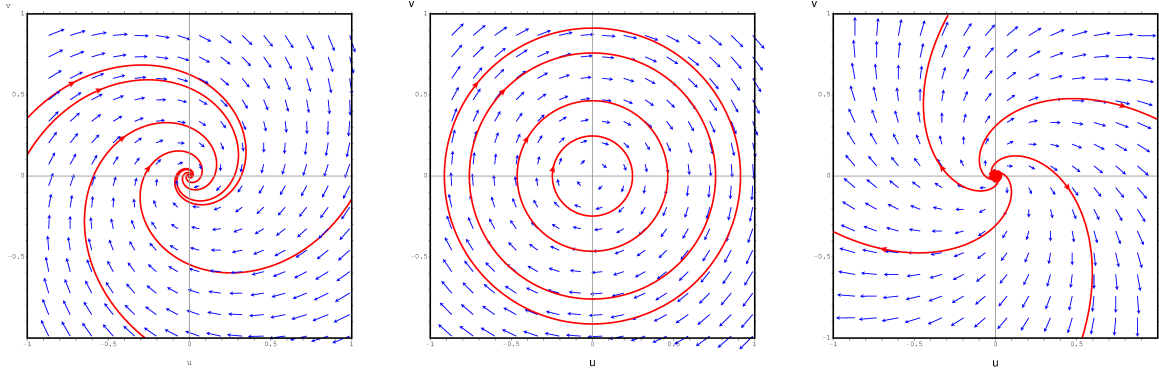


Figure 3.3: Different cases of oscillations around the origin according to (3.26) with $b = 1$: stable spiral (left, $a = -1$), center (middle, $a = 0$), and unstable spiral (right, $a = 1$).

solutions are given by

$$\xi_1(t) = e^{at} r_0 \cos(\theta_0 - bt), \quad \xi_2(t) = e^{at} r_0 \sin(\theta_0 - bt). \quad (3.31)$$

Using the trigonometric relations

$$\begin{aligned} \cos(\theta_0 - \beta t) &= \cos(\theta_0) \cos(-\beta t) + \sin(\theta_0) \sin(-\beta t) \\ &= \cos(\theta_0) \cos(\beta t) - \sin(\theta_0) \sin(\beta t) \\ \sin(\theta_0 - \beta t) &= \sin(\theta_0) \cos(-\beta t) - \cos(\theta_0) \sin(-\beta t) \\ &= \sin(\theta_0) \cos(\beta t) + \cos(\theta_0) \sin(\beta t) \end{aligned}$$

and taking into account that

$$r_0 \cos(\theta_0) = \xi_{10}, \quad r_0 \sin(\theta_0) = \xi_{20},$$

it follows that $\xi(t)$ can be written in compact form as

$$\xi(t) = e^{at} \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix} \xi_0. \quad (3.32)$$

In consequence, the solution for $x(t)$ can be written again in the form

$$x(t) = T \xi(t) = e^{at} N(t) x_0, \quad N(t) = T \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix} T^{-1}$$

with the bounded matrix $N(t)$. Accordingly, the exponential stability of the origin is ensured for $\alpha = \Re(\lambda) < 0$.

Note that the three cases discussed so far are the only ones which occur in linear systems, except the case of a zero real part eigenvalue. In this case, the associated eigenspace is the nullspace (or *kernel*)

of the matrix A , and the associated components remain of constant distance to the origin with respect to the directions defined by the nullspace (cp. example (2.19)).

Later, we will relate nonlinear flows in neighborhoods of equilibrium points in two-dimensional systems to these linear cases, in order to gain a comprehension of the effect of parameter variations on the qualitative solution behavior. Thus it is very important that the concepts and results of this section have been well understood.

To end this section, we will formally connect the preceding study with the concept of stability, and present a short systematic classification of planar flows of linear systems.

From the solutions of the different cases studied above, it is clear that all of them can be written in the form

$$\mathbf{x}(t) = e^{-\lambda t} N(t) \mathbf{x}_0, \quad -\lambda = \max_{i=1, \dots, n} \Re(\lambda_i)$$

for some bounded matrix-valued function $N(t)$, and thus for $\Re(\lambda_i) < 0$, $i = 1, \dots, n$ it follows that $\mathbf{x}(t) \rightarrow 0$. This is summarized in the following theorem.

Theorem 3.1

The origin of the linear system (3.2) is stable if all eigenvalues λ_i , $i = 1, \dots, n$ have non-positive real part, i.e. $\Re(\lambda_i) \leq 0$. If all eigenvalues have strictly negative real part, i.e. $\Re(\lambda_i) < 0$, then the origin is exponentially stable, and (2.16) holds with $\lambda = \min |\lambda_i|$, $i = 1, \dots, n$.

In the case of second order linear systems, the classification of the dynamical behavior is particularly systematic. To see this, consider the general matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (3.33)$$

with characteristic equation

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0, \quad \text{tr}(A) = a_{11} + a_{22}, \quad \det(A) = a_{11}a_{22} - a_{12}a_{21},$$

where $\text{tr}(A)$ is the trace, and $\det(A)$ the determinant of the matrix A . The eigenvalues of A are thus given by

$$\lambda_{1,2} = \frac{1}{2} \left(\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4 \det(A)} \right). \quad (3.34)$$

Thus, the dynamic behavior can be classified as depicted qualitatively in Figure 3.4, namely: if $\text{tr}(A) < 0$ (or $\text{tr}(A) > 0$), the origin is a stable (or unstable) node ($\text{tr}(A)^2 \geq 4 \det(A)$), or spiral ($\text{tr}(A)^2 < 4 \det(A)$). In the cases where $\text{tr}(A)^2 = 4 \det(A)$ there are generalized eigenvectors, and the origin is stable (or unstable) if $\text{tr}(A) < 0$ (or > 0). Note that this case represents the transition between a node and a spiral. If $\det(A) < 0$, there is always one positive and one negative eigenvalue, so that the origin is a saddle.

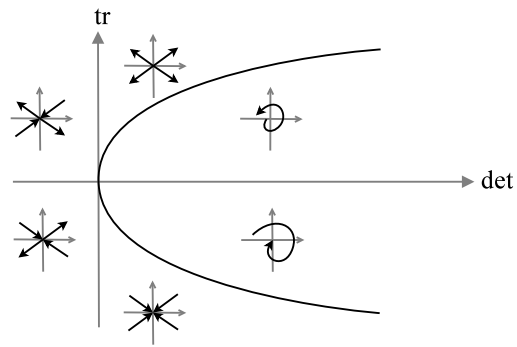


Figure 3.4: Classification of planar linear dynamics according to (3.34), with tr being the trace, and $\det$ the determinant of the dynamics matrix A .

3.2 Linear Time Varying (LTV) Systems and Floquet theory

In the sequel we consider the linear time-varying systems of the form

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0. \quad (3.35)$$

At the first place we discuss the general case for arbitrary $A(t)$, and then focuss on the particular case of periodic matrices, i.e. such that $\exists T > 0 : A(t + T) = A(t)$.

3.2.1 General LTV Systems - Fundamental matrices

There is much to say about linear time-varying systems. Actually, the subject is non trivial, as the complete system behavior heavily depends on time varying properties. This implies that the simple solution based on the matrix exponential does not apply, and the decomposition into linear subspaces associated to eigenvectors changes with time. The graphical method by looking at the vector field also becomes impractical. Hence, another approach has to be employed here.

The basic idea for analyzing and solving time-varying differential equations is by searching similarity transformations of the kind

$$\mathbf{x}(t) = P(t)\boldsymbol{\xi}(t), \quad (3.36)$$

where $P(t)$ is non-singular at any time t . Clearly,

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) = A(t)P(t)\boldsymbol{\xi}(t) = \dot{P}(t)\boldsymbol{\xi}(t) + P(t)\dot{\boldsymbol{\xi}}(t)$$

so that

$$\dot{\boldsymbol{\xi}}(t) = P^{-1}(t) (A(t)P(t) - \dot{P}(t)) \boldsymbol{\xi}(t), \quad \boldsymbol{\xi}(t_0) = P^{-1}(t_0)\mathbf{x}_0 \quad (3.37)$$

Now, $P(t)$ is called a *fundamental matrix* [Kai80], if it is a non-singular solution of the matrix differential equation

$$\dot{P}(t) = A(t)P(t), \quad P(t_0) \neq 0. \quad (3.38)$$

If P in (3.37) is a fundamental matrix, then the solution of (3.37) is

$$\boldsymbol{\xi}(t) = \boldsymbol{\xi}_0 = P^{-1}(t_0)\mathbf{x}_0$$

and the solution $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = P(t)\boldsymbol{\xi}(t) = P(t)P^{-1}(t_0)\mathbf{x}_0. \quad (3.39)$$

This gives rise to the notion of the *state transition matrix* [Kai80]

$$\varphi_t(t_0) = P(t)P^{-1}(t_0). \quad (3.40)$$

Note that, accordingly, the flow of the system is given by

$$\phi_t(t_0, \mathbf{x}_0) = \varphi_t(t_0) \mathbf{x}_0. \quad (3.41)$$

In the particular case of time-invariant systems, the state transition matrix $\varphi_t(t_0)$ is just the fundamental solution $\varphi_t = \exp(At)$. Nevertheless, if A varies over time, then the function $\exp(\int_0^t A(\tau) d\tau)$ is not a fundamental matrix, given that

$$\frac{d}{dt} \exp\left(\int_0^t A(\tau) d\tau\right) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d}{dt} \left(\int_0^t A(\tau) d\tau\right)^i$$

and, as in general $A(t)$ does not commute with its integral,

$$\frac{d}{dt} \left(\int_0^t A(\tau) d\tau\right)^2 = A \int_0^t A(\tau) d\tau + \int_0^t A(\tau) d\tau A \neq 2A \int_0^t A(\tau) d\tau$$

and the same is true for higher order terms, implying that

$$\frac{d}{dt} \exp\left(\int_0^t A(\tau) d\tau\right) \neq A(t) \exp\left(\int_0^t A(\tau) d\tau\right).$$

Once found a fundamental matrix $P(t)$ (i.e., satisfying (3.38)), the linear system can be transformed into an arbitrary linear time-invariant one. Take any matrix $\bar{A}$, and consider the state transformation

$$\mathbf{x}(t) = \bar{P}(t) \boldsymbol{\xi}(t) = P(t) e^{-\bar{A}t} \boldsymbol{\xi}(t). \quad (3.42)$$

By straightforward calculations one obtains

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t) \mathbf{x}(t) \\ &= A(t) P(t) e^{-\bar{A}t} \boldsymbol{\xi}(t) \\ &= \frac{d}{dt} \left(P(t) e^{-\bar{A}t} \boldsymbol{\xi}(t) \right) \\ &= \dot{P}(t) e^{-\bar{A}t} \boldsymbol{\xi}(t) - P(t) \bar{A} e^{-\bar{A}t} \boldsymbol{\xi}(t) + P(t) e^{-\bar{A}t} \dot{\boldsymbol{\xi}}(t) \\ &= A(t) P(t) e^{-\bar{A}t} \boldsymbol{\xi}(t) - P(t) \bar{A} e^{-\bar{A}t} \boldsymbol{\xi}(t) + P(t) e^{-\bar{A}t} \dot{\boldsymbol{\xi}}(t) \end{aligned}$$

and thus

$$\begin{aligned} \dot{\boldsymbol{\xi}}(t) &= e^{\bar{A}t} P^{-1}(t) \left(A(t) P(t) e^{-\bar{A}t} - A(t) P(t) e^{-\bar{A}t} + P(t) \bar{A} e^{-\bar{A}t} \right) \boldsymbol{\xi} \\ &= e^{\bar{A}t} \bar{A} e^{-\bar{A}t} \boldsymbol{\xi} \end{aligned}$$

and, as $\bar{A}$ and $\exp(\bar{A})$ commute (the reader should show this!), the equivalent time-invariant system dynamics are given by

$$\dot{\boldsymbol{\xi}}(t) = \bar{A} \boldsymbol{\xi}(t), \quad \boldsymbol{\xi}(t_0) = e^{\bar{A}t_0} P^{-1}(t_0) \mathbf{x}_0. \quad (3.43)$$

For $\bar{A} = 0$ this corresponds to the case discussed above. The intrinsic reason why, given a fundamental matrix $P(t)$ the system can be transformed into any time-invariant one, consists in the fact that the solution is already determined by the fundamental matrix $P(t)$. The great deal about LTV systems thus resides in solving the matrix differential equation (3.38).

3.2.2 Periodic LTV Systems - Floquet theory

Of particular interest in time-varying systems theory is the case of periodic matrixes $A(t)$, i.e. when there exists a time $T > 0$, such that

$$A(t + T) = A(t), \quad \forall t \geq 0. \quad (3.44)$$

These kind of systems has been intensively studied, initiating with the great french mathematician Gaston Floquet, and the associated theory is named after him [**Floquet**; Kai80].

First of all, note that for a periodic linear time-varying system the vector field behaves periodically. This implies that the solutions are invariant under time-shifts of exactly one period, i.e. letting the initial condition $\mathbf{x}_0$ be set at $t = t_0$, with solution trajectory $\mathbf{x}_1(t) = \mathbf{x}(t; \mathbf{x}_0, t_0)$, and letting the same initial condition be set at $t = t_0 + T$, with solution trajectory $\mathbf{x}_2(t + T) = \mathbf{x}(t + T; \mathbf{x}_0, t_0 + T)$, these two solution will be identical, i.e.

$$\mathbf{x}_1(t) = \mathbf{x}(t; \mathbf{x}_0, t_0) \equiv \mathbf{x}(t + T; \mathbf{x}_0, t_0 + T) = \mathbf{x}_2(t + T).$$

Let $P(t)$ be the fundamental matrix associated to the LTV system, then in virtue of (3.39) it follows that

$$\mathbf{x}_1(t) = P(t)P^{-1}(t_0)\mathbf{x}_0 = P(t + T)P^{-1}(t_0 + T)\mathbf{x}_0.$$

This implies that

$$P(t)P^{-1}(t_0) = P(t + T)P^{-1}(t_0 + T)$$

or equivalently

$$P(t + T) = P(t)M, \quad M = P^{-1}(t_0)P(t_0 + T) \quad (3.45)$$

where the constant matrix M is called the *monodromy matrix*⁵, which can be expressed as

$$M = e^{TB} \quad (3.46)$$

for some matrix B . Combining (3.45) with (3.41) it follows that

$$\mathbf{x}(t_0 + T) = \varphi_{t_0+T}(t_0)\mathbf{x}_0 = P(t_0 + T)P^{-1}(t_0)\mathbf{x}_0 = P(t_0)MP^{-1}(t_0)\mathbf{x}_0. \quad (3.47)$$

By the particular choice $P(t_0) = I$ (3.47) implies that

$$\mathbf{x}(t_0 + T) = M\mathbf{x}(t_0). \quad (3.48)$$

⁵From the greek *monos*, meaning one or single, and *dromos* denoting the avenue or passage.

The eigenvalues of M are called the *Floquet (or characteristic) multipliers*. If the matrix $\ln(M)$ exists⁶, then the eigenvalues of the matrix

$$B = \frac{1}{T} \ln(M) \quad (3.49)$$

are called the *Floquet (or characteristic) exponents*. The real parts of the Floquet exponents are called the *Lyapunov exponents*. Having (3.48) in mind, the following theorem relates the stability of the origin $\mathbf{x} = \mathbf{0}$ of the time-varying system (3.35) with periodic matrix $A(t)$ to the Floquet multipliers and Lyapunov exponents.

Theorem 3.2

Consider the dynamics (3.35) with periodic matrix $A(t + T) = A(t)$ for some positive $T \in \mathbb{R}_+$. The origin $\mathbf{x} = \mathbf{0}$ is asymptotically stable if all Floquet multipliers lie inside the unit circle, or equivalently, all Lyapunov exponents are negative.

Proof. Denote by μ_i , $i = 1, \dots, n$ the i -th Floquet multiplier of the monodromy matrix M defined in (3.45). As $P(t_0)$ is arbitrary, set $P(t_0) = I$ so that from (3.48) it holds that

$$\mathbf{x}(t_0 + kT) = M^k \mathbf{x}(t_0)$$

and further

$$\|\mathbf{x}(t_0 + kT)\| = \|M^k \mathbf{x}(t_0)\| \leq \|M\|^k \|\mathbf{x}(t_0)\| \leq \max_i |\mu_i|^k \|\mathbf{x}(t_0)\|.$$

Note that for any $t \geq T$ there exists a $t_0 \geq 0$ so that $t = t_0 + kT$ for some $k \in \mathbb{N}$ and thus

$$\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = \lim_{k \rightarrow \infty} \|\mathbf{x}(t_0 + kT)\| \leq \lim_{k \rightarrow \infty} \max_i |\mu_i|^k \|\mathbf{x}(t_0)\|.$$

By assumption it holds that $\max_i |\mu_i| < 1$ and thus $\lim_{k \rightarrow \infty} \max_i |\mu_i|^k = 0$, implying the result. $\square$

Note that by definition the monodromy matrix in (3.45) depends on the initial time t_0 . Nevertheless, the Floquet multipliers do not depend on $P(t_0)$. This becomes clear when considering two fundamental matrixes $P_1(t)$ and $P_2(t)$ and thus two associated monodromy matrices M_1 and M_2 . As both satisfy (3.38) there must be a constant (non-singular) matrix S such that $P_2 = P_1 S$, and consequently the associated monodromy matrices M_1 and M_2 satisfy⁷

$$M_1 = P_1^{-1}(t_0)P_1(T) = [P_2(t_0)S^{-1}]^{-1}[P_2(T)S^{-1}] = S[P_2^{-1}(t_0)P_2(T)]S^{-1} = SM_2S^{-1}.$$

This, in turn, implies that both monodromy matrixes are similar, and thus have the same eigenvalues.

The main result of the Floquet theory is stated next.

⁶A treatment on the matrix logarithm is not subject of the present lecture notes, but it is well known from the scalar case that $\ln(\lambda)$ does only exist for $\lambda \neq 0$. For a matrix the logarithm is normally determined using previous diagonalization or transformation into Jordan normal form. Note also that in the case of complex eigenvalues, the matrix logarithm may be non-unique.

⁷In the third step, the relation for the matrix inverse $(AB)^{-1} = B^{-1}A^{-1}$ is used, which holds if A and B are invertible and of appropriate dimensions. As an exercise, the reader should convince himself that this relation is true.

Theorem 3.3

Let M be so that $\ln(M)$ exists. Then there exists a T -periodic matrix-valued function $\Psi(t)$ and a constant matrix $B = B(t_0)$, such that

$$P(t) = \Psi(t)e^{tB}, \quad B = \frac{1}{T}\ln(M) \quad (3.50)$$

with the monodromy matrix $M = M(t_0)$ defined in (3.45).

The proof of the theorem is quite simple, having the above facts on the monodromy matrix M as points of departure. Actually, notice that, given $P(t)$, the matrix Ψ can always be determined from (3.50), and satisfies (taking into account the definition (3.46))

$$\begin{aligned} \Psi(t+T) &= P(t+T)e^{-(t+T)B} = P(t+T)e^{-TB}e^{-tB} \\ &= P(t)Me^{-TB}e^{-tB} = P(t)MM^{-1}e^{-tB} = P(t)e^{-tB} = \Psi(t). \end{aligned}$$

The importance of this theorem consists in that it allows to decompose the solution of a periodic time-variant system into a periodic part (Ψ), and an exponential part (e^{tB}). This is particularly useful when analyzing the stability of periodic orbits of linear and nonlinear systems, as will be discussed later. For the moment, notice that with (3.50), the proof of Theorem 3.2 could be reformulated using

$$\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| \leq \lim_{t \rightarrow \infty} \|\Psi(t)\| \|e^{tB}\| \|P^{-1}(t_0)\| \|\mathbf{x}_0\|,$$

and is left to the reader as an exercise.

In the light of the representation (3.50), the Floquet decomposition ensures that there exists a coordinate $\xi(t) = \Psi(t)^{-1}\mathbf{x}(t)$ with dynamics (cp. (3.43))

$$\dot{\xi} = B\xi, \quad \xi(t_0) = \Psi^{-1}(t_0)\mathbf{x}_0, \quad \Leftrightarrow \quad \xi(t) = e^{tB}\xi(t_0) \quad (3.51)$$

underlining the fact that the stability properties are determined by the Floquet exponents, i.e. the eigenvalues of B , as stated in Theorem 3.2. Actually the solutions $\mathbf{x}(t)$ satisfy

$$\mathbf{x}(t) = \Psi(t)e^{tB}\Psi^{-1}(t_0)\mathbf{x}_0,$$

and with $\lim_{t \rightarrow \infty} e^{tB} = 0$ it follows that $\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = 0$.

The usefulness of these results will be illustrated later, when dealing with periodic orbits of nonlinear systems, in particular for their stability analysis. The analytic calculation of the Floquet exponents σ_i , $i = 1, \dots, n$ (or multipliers) is in most cases a rather complex task and thus the Floquet exponents are normally calculated using numerical tools. For a numerical determination of the Floquet multipliers and exponents, it is sufficient to solve the matrix differential equation (3.38) with initial condition $P(t_0) = I$. In virtue of (3.45), in this case, the Floquet multipliers are just the eigenvalues of $M = P(T)$, and the Floquet exponents are the eigenvalues of the matrix

$$B = \frac{1}{T}\ln(M) = \frac{1}{T}\ln[P(T)]. \quad (3.52)$$

To illustrate the concepts presented in this section, consider the following dynamics

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} a + \cos(t) & c \\ 0 & b - \sin(t) \end{bmatrix}}_{A(t)} \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (3.53)$$

Clearly, A is periodic with period $T = 2\pi$. First, the fundamental matrix $P(t)$ has to be determined, by solving the matrix differential equation (3.38) with initial condition $P(t_0) = I$, i.e.

$$\begin{aligned} \dot{P}(t) &= A(t)P(t) = \begin{bmatrix} a + \cos(t) & c \\ 0 & b - \sin(t) \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \\ &= \begin{bmatrix} (a + \cos(t))p_{11}(t) + cp_{21}(t) & (a + \cos(t))p_{12}(t) + cp_{22}(t) \\ (b - \sin(t))p_{21}(t) & (b - \sin(t))p_{22}(t) \end{bmatrix}, \quad P(t_0) = I \end{aligned}$$

or equivalently

$$\begin{aligned} \dot{p}_{11}(t) &= (a + \cos(t))p_{11}(t) + cp_{21}(t), & p_{11}(t_0) &= 1 \\ \dot{p}_{12}(t) &= (a + \cos(t))p_{12}(t) + cp_{22}(t), & p_{12}(t_0) &= 0 \\ \dot{p}_{21}(t) &= (b - \sin(t))p_{21}(t), & p_{21}(t_0) &= 0 \\ \dot{p}_{22}(t) &= (b - \sin(t))p_{22}(t), & p_{22}(t_0) &= 1 \end{aligned}$$

The solution of p_{21}, p_{22} are thus

$$\begin{aligned} p_{21}(t) &= 0, \\ p_{22}(t) &= e^{\int_{t_0}^t (b - \sin(\tau)) d\tau} = e^{b(t-t_0) + (\cos(t) - \cos(t_0))}, \end{aligned}$$

and the solutions for p_{11}, p_{12} are consequently

$$p_{11}(t) = e^{a(t-t_0) + (\sin(t) - \sin(t_0))}$$

and

$$p_{12}(t) = c \int_{t_0}^t e^{a(t-\tau-t_0) + (\sin(t-\tau) - \sin(t_0))} p_{22}(\tau) d\tau.$$

Consider the simplest case, which is given for $c = 0$. The solution for $P(t)$ then satisfies

$$\begin{aligned} P(t) &= \begin{bmatrix} e^{a(t-t_0) + (\sin(t) - \sin(t_0))} & 0 \\ 0 & e^{b(t-t_0) + (\cos(t) - \cos(t_0))} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} e^{-at_0 + \sin(t) - \sin(t_0)} & 0 \\ 0 & e^{-bt_0 + \cos(t) - \cos(t_0)} \end{bmatrix}}_{\Psi(t)} \underbrace{\begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}}_{e^{tB}} \end{aligned}$$

with the matrix B and its eigenvalues (the Floquet exponents)

$$B = \frac{1}{T} \begin{bmatrix} aT & 0 \\ 0 & bT \end{bmatrix}, \quad \sigma_1 = a, \quad \sigma_2 = b.$$

For $t_0 = 0$ the analytic solution of $\mathbf{x}$ is given by

$$\mathbf{x}(t) = P(t)\mathbf{x}_0 = \begin{bmatrix} e^{at + \sin(t)} & 0 \\ 0 & e^{bt + \cos(t) - 1} \end{bmatrix} \mathbf{x}_0$$

and the monodromy matrix is given by

$$M = P(T) = P(2\pi) = \begin{bmatrix} e^{2a\pi} & 0 \\ 0 & e^{2b\pi} \end{bmatrix}.$$

Thus the Floquet multipliers are $\mu_1 = e^{2a\pi}$, $\mu_2 = e^{2b\pi}$. This corresponds with the Floquet exponents calculated above, which in this case are real and thus correspond also to the Lyapunov exponents. According to Theorem 3.2, the origin $\mathbf{x} = \mathbf{0}$ is asymptotically stable for $a, b < 0$.

Figure 3.5 shows on the left hand side the time response of (3.53) for $a = -0.2, b = -0.5$ and initial condition $\mathbf{x}_0 = [1 \ 1]^T$, and on the right hand side the phase plane for different initial conditions. One appreciates the asymptotic convergence with settling time $t_s = 4t_c = 4 \frac{1}{0.2} = 20$ time units, according with the relation (2.18).

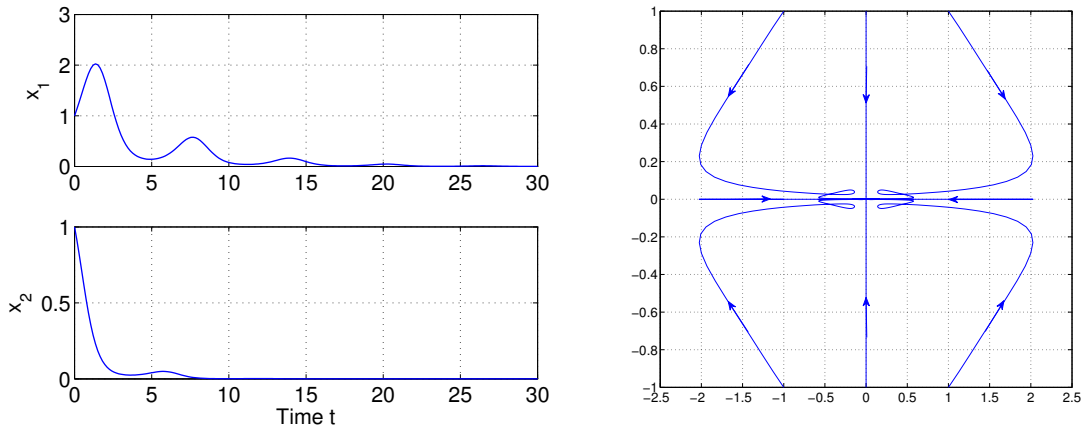


Figure 3.5: Left: Time response of the dynamics (3.53) for $a = -0.2, b = -0.5$ and $\mathbf{x}_0 = [1, 1]^T$. Right: The associated phase portrait for different initial conditions.

This example is also useful to illustrate the following fact. The time-varying eigenvalues of A are given by $\lambda_1 = a + \cos(t)$ and $\lambda_2 = b - \sin(t)$, showing that for $|a|, |b| < 1$, as in the case example presented in Figure 3.5, they will change sign, so that no classical argument on the relation between the sign of the eigenvalues and the stability of the origin would apply. Nevertheless, as long as $a, b < 0$, the origin is asymptotically stable. This exemplifies the fact that the stability of time-varying systems can not be analyzed on the basis of an eigenvalue approach of the dynamics matrix A , but needs to be discussed using the fundamental matrix P , or the Floquet multipliers (if A is periodic), or some other methods (like Lyapunov theory⁸).

References

- [Fis09] G. Fischer. *Lineare Algebra*. Vieweg, 2009 (cit. on pp. 16, 17).
- [Kai80] T. Kailath. *Linear Systems*. Prentice Hall, Inc., 1980 (cit. on pp. 27, 29).

⁸Compare Section 4.3.

Nonlinear flows - An introduction

In this chapter the basic theory of nonlinear flows is introduced. In contrast to linear systems with only one equilibrium, a continuum set of equilibria, or a continuum family of periodic orbits, in nonlinear flows the local behavior close to equilibrium can not be extended to the global behavior, i.e. even though an equilibrium point may be stable, there may happen much more things far from this equilibrium point. Examples are the presence of multiple attractors, each of which has a domain of attraction. Additionally, these attractor sets may be isolated periodic orbits, enclosing one or more equilibria. Hence it makes sense to distinguish in nonlinear systems between local and non-local behavior, i.e. close and far from equilibrium, respectively. This distinction motivates the structure of the present chapter, where at first place, the main results of local theory are presented, and afterwards, non-local theory is addressed.

Some words on the terminology and notion of locality are in order here. From a stability analysis point of view, in the sense of Lyapunov stability, local normally refers to a given set in state space, which may be small or large. Global, in contrast, refers to properties which are valid in the whole $\mathbb{R}^n$. From a practical point of view, such a globality is almost never of interest, as any physical system exists within some bounded region of possible states (although it is not always modeled explicitly with these restrictions). A simple example is a pendulum, which obviously has a limited velocity from a practical point of view. Actually, a possible limit velocity is given by the maximum centrifugal force the articulation supports. There does not exist any articulation supporting infinite velocity... Another classical example are reacting systems where concentrations are always bounded between 0% and 100%. There is many more to say about this subject, but this is not the place to do so. Nevertheless, it should be clear that from a practical point of view, the state space is always restricted.

From a nonlinear dynamics point of view, local refers much more to the behavior of trajectories close to equilibrium, i.e. in a non-defined, sufficiently small neighborhood. This concept is straightly related to the Taylor approximation of the vector field around the equilibrium by small (first to third) order terms. Clearly, the third order Taylor approximation does not reveal anything certain when fourth order terms dominate the first-to-third order ones. Close to the equilibrium, nevertheless, it is clear that the behavior will be qualitatively governed by the first-to-third order terms (unless they vanish). This is why we talk about local behavior. The contrary of local behavior could now be called global behavior, or non-local behavior. Here, the term non-local is preferred, in order to avoid the above-mentioned well accepted globality concept handled in Lyapunov stability theory.

4.1 Local Theory

In this section the basic concepts and results from local theory for nonlinear systems are presented, i.e. concerning the behavior of nonlinear flows close to equilibrium. The main results on this subject are the Hartman-Grobman Theorem, which relates the behavior of the nonlinear system to its linear first-order Taylor approximation, and the Center Manifold Theorem, which applies to nonlinear flows which can not be handled with first-order terms only.

4.1.1 The Hartman-Grobman Theorem

In many cases of interest, the qualitative behavior of the nonlinear system (2.2) can be determined via the consideration of the dynamics close to the equilibrium $\mathbf{x}^*$ (i.e., such that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$). For this purpose, consider the difference

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}^*$$

with dynamics

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \mathbf{f}(\mathbf{x}) - \underbrace{\mathbf{f}(\mathbf{x}^*)}_{=0} = \mathbf{f}(\mathbf{x}^* + \tilde{\mathbf{x}}) - \mathbf{f}(\mathbf{x}^*) \\ &= \mathbf{f}(\mathbf{x}^*) + \frac{\partial \mathbf{f}(\mathbf{x}^*)}{\partial \mathbf{x}} \tilde{\mathbf{x}} + \mathcal{O}^2(\tilde{\mathbf{x}}) - \mathbf{f}(\mathbf{x}^*) \\ &\approx \frac{\partial \mathbf{f}(\mathbf{x}^*)}{\partial \mathbf{x}} \tilde{\mathbf{x}}\end{aligned}$$

for $\mathcal{O}^2(\tilde{\mathbf{x}}) \approx \mathbf{0}$, i.e. in a sufficiently small neighborhood of $\mathbf{x}^*$. This seems to imply that the dynamics close to the equilibrium point are in fact approximated by the so-called *linearization*

$$\dot{\mathbf{x}} = J(\mathbf{x}^*, \mathbf{p})\mathbf{x}, \quad J(\mathbf{x}^*, \mathbf{p}) = \frac{\partial \mathbf{f}(\mathbf{x}^*, \mathbf{p})}{\partial \mathbf{x}}, \quad (4.1)$$

of the nonlinear dynamics around this equilibrium where $J(\mathbf{x}^*, \mathbf{p})$ is the Jacobian matrix associated to the function $\mathbf{f}(\mathbf{x}, \mathbf{p})$ evaluated at the pair $(\mathbf{x}^*, \mathbf{p})$. Nevertheless, in nonlinear systems some caution has to be taken when considering linear approximations, as becomes clear when considering the dynamics

$$\dot{x} = f(x) = -x^3, \quad x(0) = x_0, \quad \frac{\partial f(0)}{\partial x} = 0. \quad (4.2)$$

The linearization implies that $x(t) = x_0$ remains constant, but the origin actually is an asymptotically stable equilibrium point, attracting globally all trajectories, as can be easily verified by plotting $\dot{x}$ over x (see Figure 4.1)

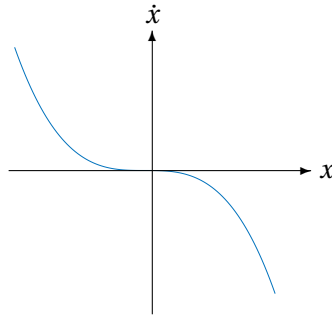


Figure 4.1: Diagram of $\dot{x}$ versus x for (4.2) associated to the asymptotically stable equilibrium point $x = 0$.

In the sequel, whenever it is not necessary to explicitly indicate the parameter vector $\mathbf{p}$ the jacobian is written simply as $J(\mathbf{x}^*)$.

Clearly, studying the flow associated to the linear system (4.1) is much easier than evaluating the flow of the nonlinear system. The following theorem, due to Hartman and Grobman, establishes sufficient conditions for the qualitative equivalence between the trajectories of (2.2) and (4.1) within a sufficiently small neighborhood of $(\mathbf{x}^*)$. In order to state the theorem, recall the definition of a homeomorphism $\mathbf{h}$ (or topological isomorphism) between two vector (more exactly topological)

spaces as a continuous map with continuous inverse. Homeomorphisms are such that they conserve properties like closeness of two points (due to the continuity) in both spaces, and are frequently employed in the analysis of dynamical systems. Simple examples are coordinate-transformations (e.g. from cartesian to polar coordinates, and vice versa). Differentiable homeomorphisms are called diffeomorphisms, and are a useful tool in nonlinear control theory, in particular for feedback linearization, flatness analysis, observability analysis, and others (see e.g. [Isi95; Sas99]).

On the basis of homeomorphisms it is possible to introduce the following concept, comparing the flow (i.e., the solution) of two different dynamical systems.

Definition 4.1

The flows of the dynamical systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p})$, $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\boldsymbol{\xi}} = \mathbf{g}(\boldsymbol{\xi}, \mathbf{p})$, $\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0$ are said to be topologically equivalent if there exists a homeomorphism $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, such that $\boldsymbol{\xi}(t) = \mathbf{h}(\mathbf{x}(t))$ for all $t \in [0, T]$.

Typical examples of topological equivalence are the ones we studied in the preceding chapter when solving linear differential equations using state transformations into diagonal, or (real) Jordan normal form, i.e. using similarity transformations $T \in \mathcal{O}(\mathbb{R}^n)$ such that $\boldsymbol{\xi} = \mathbf{h}(\mathbf{x}) = T^{-1}\mathbf{x}$. Clearly, when a dynamical system in a complex form is topologically equivalent to a dynamical system which has a quite simple form, then the behavior of the dynamics can be studied on the basis of the simpler representation. This fact was what we used when we talked about linear dynamical systems and their behavior.

The next definition concerns a property of the eigenvalues of an equilibrium point is quite useful to characterize equilibria.

Definition 4.2

An equilibrium point $\mathbf{x}^*$ is called hyperbolic if the jacobian $J(\mathbf{x}^*)$ (4.1) has no eigenvalues on the imaginary axis.

Having these concepts at hand the following result can be stated.

Theorem 4.1

If $\mathbf{x}^*$ is a hyperbolic equilibrium point of the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}) \quad (4.3)$$

then there exists an open neighborhood $\mathcal{N}(\mathbf{x}^*)$ of $\mathbf{x}^*$, and a homeomorphism $\mathbf{h} : \mathcal{N}(\mathbf{x}^*) \rightarrow \mathbb{R}^n$, such that $\mathbf{h}(\mathbf{x}^*) = \mathbf{0}$ and

$$\mathbf{x}(t) = \boldsymbol{\phi}_t(\mathbf{x}_0, \mathbf{p}) = \mathbf{h}^{-1} \left(e^{J(\mathbf{x}^*, \mathbf{p})t} \mathbf{h}(\mathbf{x}_0) \right), \quad (4.4)$$

where $\boldsymbol{\phi}_t(\mathbf{x}_0, \mathbf{p})$ is the flow associated to (4.3) at time t , i.e. the flow of (2.2) is locally (in $\mathcal{N}(\mathbf{x}^*)$) topologically equivalent to the flow of the linearization (4.1).

The proof of this theorem goes beyond the scope of the present manuscript, but can be found in specialized literature on this subject (e.g. [Har64; Har60a; Har60b; Per01]).

It is important to note, that the Hartman-Grobman theorem only applies for hyperbolic equilibria, i.e. those whose associated Jacobian has no eigenvalues on the imaginary axis. The case of eigenvalues on the imaginary axis will be discussed in Section 4.1.2. In example (4.2) the flow of the linearization and the nonlinear system are not topologically equivalent, given that in the linearization all solutions remain constant and will thus be mapped by any homeomorphism to the same point in state space, while for the nonlinear system these points move with time.

For nonlinear systems, the linearization $J(\mathbf{x}^*, \mathbf{p})$ is often the first access to establish the qualitative behavior of the flow $\phi_t(\mathbf{x}_0)$, and is often sufficient for a local (i.e., close to the equilibrium point) qualitative analysis. The next subsection contains some examples to illustrate this approach.

4.1.1.1 Examples

In order to illustrate the concept of topological equivalence and the localness property announced in the Hartman-Grobman theorem 4.1, some case examples are in order.

The unforced Duffing equation

The unforced Duffing equation is frequently used to study damped driven nonlinear oscillators and is given by

$$\ddot{x} + \delta \dot{x} - x + x^3 = 0 \quad (4.5)$$

In state space form, with $x_1 = x, x_2 = \dot{x}$ this equation is written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 - \delta x_2 \end{aligned} \quad (4.6)$$

Clearly, the SSs of this system are given by $x_2^* = 0$ and $x_1^* = \pm\sqrt{1}$, or $x_1^* = 0$. The Jacobian of the system at a point $\mathbf{x}^* = [x_1^* \ x_2^*]^T$ is given by

$$J[\mathbf{x}^*] = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^{*2} & -\delta \end{bmatrix}$$

and the associated eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2} \left(-\delta \pm \sqrt{\delta^2 + 4 - 12x_1^{*2}} \right)$$

Accordingly, the SS $\mathbf{x}^* = \mathbf{0}$ is a saddle point for every value of δ . Note that in the change from $\delta < 0$ to $\delta > 0$ the stable and unstable directions of the SS interchange. The other two SSs can be analyzed equivalently, due to the symmetry feature of the dynamics with respect to the x_1 -coordinate. The corresponding eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2} \left(-\delta \pm \sqrt{\delta^2 - 8} \right)$$

implying that the SS locally corresponds to

- an unstable node for $\delta \leq -\sqrt{8}$
- an unstable spiral for $-\sqrt{8} < \delta < 0$

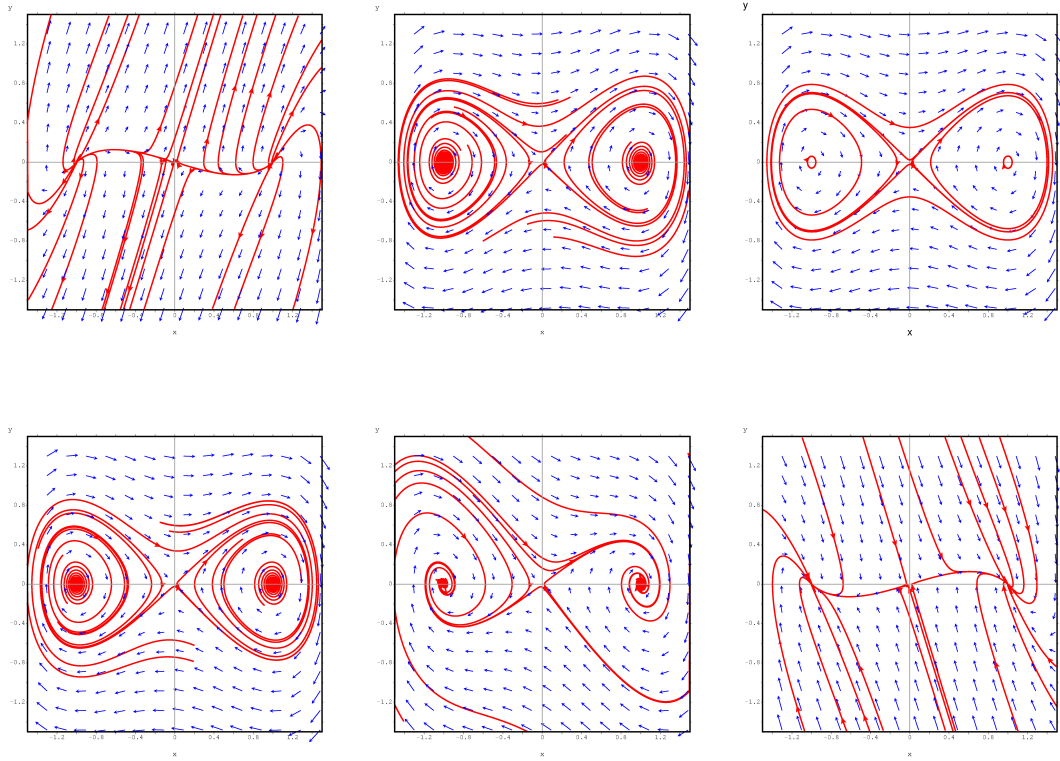


Figure 4.2: Phase plane of the Duffing equation (4.5) for: $d = -3 \approx -\sqrt{8}$ (upper left: a saddle with two repulsor nodes having almost generalized eigenvectors), $d = -0.1$ (upper middle: a saddle with two unstable spirals), $d = 0$ (upper right, homoclinic bifurcation: a saddle with homoclinic trajectories, and two neighboring centers), $d = 0.1$ (lower left: a saddle with two attractor spirals), $d = 0.6$ (lower middle: a saddle with two neighboring attractor spirals), and $d = 3 \approx \sqrt{8}$ (lower right: a saddle with two attractor nodes having almost generalized eigenvectors).

- a stable spiral for $0 < \delta < \sqrt{8}$
- a stable node for $\sqrt{8} \leq \delta$.

At $\delta = \pm\sqrt{8}$ the eigenvalues have multiplicity two, and generalized eigenvectors. Recall that the generalized eigenvectors present the transition between a node and a spiral.

For $\delta = 0$ the eigenvalues are purely imaginary, predicting a center, but no conclusion can be taken using Hartman-Grobmann's criterion, because the SS is not hyperbolic ($Re(\lambda_i) = 0$, $i = 1, 2$).

This analysis characterizes the local behavior, close to the SSs. It does not explain what phenomena can occur away from the SS. Given that for $\delta > 0$ the stabilizing linear and nonlinear terms dominate the linear destabilizing ones far away from equilibrium, it is clear that the trajectories are bounded. Thus, given that trajectories can not cross each other, the saddle and spiral have to be connected in some way. So far, the non-local behavior has to be analyzed using numerical integration tools, and some illustrating cases are shown in Figure 4.2, verifying the local analysis presented here.

A nonlinear spiral

Consider the following dynamics

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}\tag{4.7}$$

Clearly, $\mathbf{x}^* = \mathbf{0}$ is an equilibrium point. It is not quite simple to see if there are further equilibria for (4.7), but for the moment consider only the origin. The associated Jacobian matrix is given by

$$J[\mathbf{0}] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and has eigenvalues

$$\lambda_{1,2} = 1 \pm i$$

predicting that the origin is locally equivalent to an unstable linear spiral. This is the only thing the Hartman-Grobman theorem predicts.

To see what happens when x grows spiraling outwards, and in particular to revise if there exist other equilibria, consider the change to polar coordinates

$$\begin{aligned} \dot{r} &= r(1 - r^2) \\ \dot{\theta} &= 1. \end{aligned} \tag{4.8}$$

This clearly shows that: (i) the only equilibrium point is given by $\mathbf{x}^* = \mathbf{0}$, (ii) trajectories continue oscillating for all times with constant frequency 1, and associated period $T = 2\pi$, (iii) the radial dynamics are independent of the angular ones, and (iv) in the r -direction there is an equilibrium distance given by $r = 1$, which corresponds to a unique attractor for the r -dynamics. Thus, there must be an isolated periodic orbit at constant radius $r = 1$. This behavior is illustrated in Figure 4.3. Note that the simulation verifies the local behavior predicted by the Hartman-Grobman theorem. Isolated,

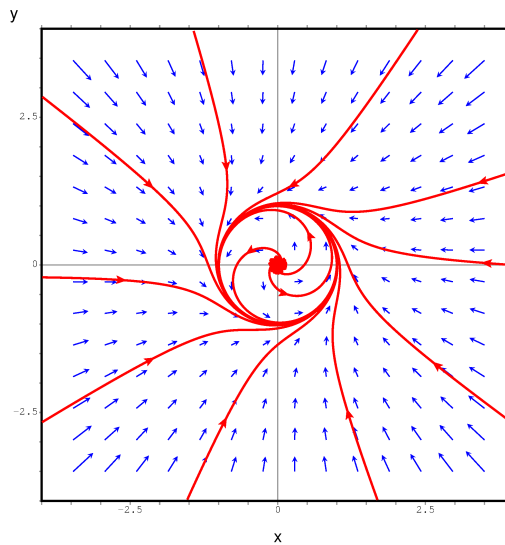


Figure 4.3: Limit cycle about an equilibrium point, with local behavior close to the equilibrium equivalent to its linearization.

closed, periodic orbits (or trajectories) are called limit cycles, and are a nonlinear phenomenon which cannot be encountered in linear systems. Linear systems may exhibit periodic orbits, but they will never be isolated, but are accompanied by a continuum family of isolated orbits with different radii. The subject of limit cycles is of paramount importance in science and engineering applications and will be studied with more details later.

4.1.2 The center-manifold theorem

As shown in the preceding section on the Hartman-Grobman theorem, an important means on the behavior assessment of nonlinear systems in a vicinity of hyperbolic equilibrium points (i.e., those with real part different to zero) is given by looking at the (Taylor) linearization. Nevertheless, a great deal of nonlinear dynamics is about non-hyperbolic problems, i.e. systems with equilibria that have eigenvalues on the imaginary axis. In these cases the Hartman-Grobman theorem does not apply, and nothing can be said about the flow near the equilibrium point looking only at the linearization. A typical example is given by (4.2)

Another example is given by the dynamics

$$\dot{x} = x^2$$

with saddle equilibrium at $x = 0$. The linearization is the same as in the preceding case ($\dot{x} = 0$), and predicts that $x = 0$ is stable. Nevertheless, for the actual nonlinear dynamics the equilibrium is unstable (a saddle) and furthermore the system has finite escape time behavior, for all positive initial conditions $x_0 > 0$, as discussed in Section 2.2.

These examples already show that interesting things happen which can not be explained using linear arguments. This explains why some experts say that the non-hyperbolic dynamics are the real nonlinear ones.

Nevertheless, considering that all hyperbolic eigenvalues lead locally to simple dynamics, the corresponding eigenvectors are tangent to stable and unstable manifolds (the so-called *inset* and *outset*). From a practical point of view it is particularly interesting to analyze stability of systems with zero eigenvalues also, in order to design them accurately, or eventually stabilize them using feedback control. Thus, assume for the moment that all eigenvalues with non-zero real part are negative, and thus, the system can be brought by a linear transformation into the form

$$\begin{aligned}\dot{\mathbf{x}}_s &= A_s \mathbf{x}_s + \mathbf{f}_s(\mathbf{x}_s, \mathbf{x}_z) \\ \dot{\mathbf{x}}_z &= A_z \mathbf{x}_z + \mathbf{f}_z(\mathbf{x}_s, \mathbf{x}_z)\end{aligned}\tag{4.9}$$

with indices s, z representing stable and center (*Zentrum*) components of the state. The matrices A_s and A_z are given by the linearization of the system at the equilibrium point $\mathbf{x}_*$, and the eigenvalues of A_z have zero real part, while the eigenvalues of A_s all have negative real part. The functions $\mathbf{f}_s$ and $\mathbf{f}_z$ satisfy

$$\mathbf{f}_s(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad \mathbf{f}_z(\mathbf{0}, \mathbf{0}) = \mathbf{0}.\tag{4.10}$$

The question is how the state vector $\mathbf{x}(t) = [\mathbf{x}_s(t)^T \quad \mathbf{x}_z(t)^T]^T$ evolves over time close to the origin $[\mathbf{x}_s, \mathbf{x}_z]^T = \mathbf{0}^T$. Given the preceding discussion on the Hartman-Grobman theorem, it is intuitive that the stable part locally converges to some manifold

$$\mathbf{x}_s = \mathbf{h}(\mathbf{x}_z)\tag{4.11}$$

(the center manifold) which reaches the origin at $\mathbf{x}_z = \mathbf{0}$ and is tangential to the $\mathbf{x}_z$ -axis there, i.e.

$$\mathbf{h}(\mathbf{0}) = \mathbf{0}, \quad \frac{\partial \mathbf{h}(\mathbf{0})}{\partial \mathbf{x}_z} = \mathbf{0}.$$

Locally, the dynamics will completely depend on the flow on this manifold $\mathbf{h}(\mathbf{x}_z)$, given by the restriction

$$\dot{\xi} = A_z \xi + \mathbf{f}_z(\xi, \mathbf{h}(\xi)) = \boldsymbol{\eta}(\xi), \quad \xi(0) = \mathbf{x}_{z0}. \quad (4.12)$$

On the other hand (4.11) implies that

$$\dot{\mathbf{x}}_s|_{\mathbf{x}_s=\mathbf{h}(\mathbf{x}_z)} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}_z} \dot{\mathbf{x}}_z = \frac{\partial \mathbf{h}}{\partial \mathbf{x}_z} [A_z \mathbf{x}_z + \mathbf{f}_z(\mathbf{x}_z, \mathbf{h}(\mathbf{x}_z))] = A_s \mathbf{h}(\mathbf{x}_z) + \mathbf{f}_s(\mathbf{h}(\mathbf{x}_z), \mathbf{x}_z).$$

Summarizing, the center manifold $\mathbf{h}(\mathbf{x}_z)$ has to satisfy the partial differential equation

$$\begin{aligned} \frac{\partial \mathbf{h}}{\partial \mathbf{x}_z} [A_z \mathbf{x}_z + \mathbf{f}_z(\mathbf{x}_z, \mathbf{h}(\mathbf{x}_z))] &= A_s \mathbf{h}(\mathbf{x}_z) + \mathbf{f}_s(\mathbf{h}(\mathbf{x}_z), \mathbf{x}_z) \\ \mathbf{h}(\mathbf{0}) &= \mathbf{0}, \quad \frac{\partial \mathbf{h}(\mathbf{0})}{\partial \mathbf{x}_z} = \mathbf{0}. \end{aligned} \quad (4.13)$$

This result is formally stated in the following theorem (cp. e.g. [GH77; Sas99]).

Theorem 4.2

Let $\mathbf{f} \in \mathcal{C}^r(X)$, with $r \geq 1$ and $X \subset \mathbb{R}^n$ an open set, $\mathbf{0} \in X$. If $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, and $J(\mathbf{0})$ has z eigenvalues with zero real part, and $s = n - z$ eigenvalues with negative real part, then the system can be brought into the diagonal-like form (4.9) with state $\mathbf{x} = [\mathbf{x}_s^T \mathbf{x}_z^T]^T$, and there exist a constant $\delta > 0$, the associated ball N_δ of radius δ around the origin, and a function $\mathbf{h} \in \mathcal{C}^r(N_\delta)$ with $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, $\frac{\partial \mathbf{h}(\mathbf{0})}{\partial \mathbf{x}_z} = \mathbf{0}$ that defines the local center manifold

$$\mathcal{W}^c = \{\mathbf{x} = [\mathbf{x}_s^T, \mathbf{x}_z^T]^T \in N_\delta | \mathbf{x}_s = \mathbf{h}(\mathbf{x}_z)\} \quad (4.14)$$

and satisfies the partial differential equation (4.13) in N_δ . The flow on $\mathcal{W}^c$ is determined by (4.12).

If the origin $\mathbf{x}_z = \mathbf{0}$ of (4.12) is (locally) asymptotically stable in the set $\mathcal{W}^c$, then it is for the system (4.9), given that, locally, $\mathcal{W}^c$ is an attractor set for the flow. This is formally reflected in the following theorem (cp. [Sas99]).

Theorem 4.3

If the dynamics (4.12) are stable, then there exists a constant $\gamma > 0$ so that locally (in $\mathcal{N}_\delta$) for t large enough the solutions of (4.9) satisfy

$$\begin{aligned} \mathbf{x}_z(t) &= \xi(t) + \mathbf{O}(e^{-\gamma t}) \\ \mathbf{x}_s(t) &= \mathbf{h}[\xi(t)] + \mathbf{O}(e^{-\gamma t}) \end{aligned} \quad (4.15)$$

with $\xi(t)$ being the solution of (4.12).

This result allows to analyze the stability properties of nonlinear systems on a lower dimensional submanifold of $\mathbb{R}^n$ (the center manifold). Unless in most cases, it is hard or even impossible to determine the exact mathematical form of the function $\mathbf{h}$ satisfying (4.13), it is possible to employ a polynomial

approximation of $\mathcal{W}^c$ (4.11) by considering a Taylor approximation $\boldsymbol{\phi}(\mathbf{x}_z) \approx \mathbf{h}(\mathbf{x}_z)$ satisfying (4.13) up to terms of order $k+1$, i.e.

$$\begin{aligned} \frac{d\boldsymbol{\phi}}{d\mathbf{x}_z} [A_z \mathbf{x}_z + \mathbf{f}_z(\mathbf{x}_z, \boldsymbol{\phi}(\mathbf{x}_z))] &= A_s \boldsymbol{\phi}(\mathbf{x}_z) + \mathbf{f}_s(\boldsymbol{\phi}(\mathbf{x}_z), \mathbf{x}_z) + \mathbf{O}^{k+1} \\ \boldsymbol{\phi}(\mathbf{0}) &= \mathbf{0}, \quad \frac{\partial \boldsymbol{\phi}(\mathbf{0})}{\partial \mathbf{x}_z} = \mathbf{0}. \end{aligned}$$

This approach is illustrated for the following example (e.g. [Per01; Sas99]):

$$\begin{aligned} \dot{x}_s &= -x_s + ax_z^2 \\ \dot{x}_z &= x_s x_z \end{aligned} \tag{4.16}$$

Consider the second order approximation

$$\boldsymbol{\phi}(x_z) = c_0 + c_1 x_z + c_2 x_z^2$$

of the center manifold. In virtue of the conditions $\boldsymbol{\phi}(0) = 0$ and $\frac{\partial \boldsymbol{\phi}(0)}{\partial x_z} = 0$ it follows that

$$c_0 = 0, \quad c_1 = 0, \quad \boldsymbol{\phi}(x_z) = c_2 x_z^2.$$

The associated equation (4.13) thus reads

$$\frac{\partial \boldsymbol{\phi}(x_z)}{\partial x_z} \dot{x}_z|_{x_s=\boldsymbol{\phi}(x_z)} = 2c_2 x_z (c_2 x_z^2) x_z = 2c_2^2 x_z^4 \stackrel{!}{=} \dot{x}_s|_{x_s=\boldsymbol{\phi}(x_z)} = -c_2 x_z^2 + ax_z^2$$

implying that for $c_2 = a$ a fourth order approximation is achieved in the sense that locally (i.e., for x_z such that $\mathcal{O}^4(x_z) \approx 0$) the center manifold is given by

$$h(x_z) \approx \boldsymbol{\phi}(x_z) = ax_z^2 \tag{4.17}$$

and the dynamics on $\boldsymbol{\phi}$ are locally (in the above sense) given by

$$\dot{x}_z = ax_z^3.$$

Accordingly, the origin is asymptotically stable for $a < 0$, and unstable for $a > 0$. This is illustrated in Figure 4.4, together with the approximation $\boldsymbol{\phi}(x_z)$ of the center manifold $h(x_z)$.

It can be appreciated in this figure, that locally (i.e. close to the origin) the center manifold is well approximated by the parabolic curve $\boldsymbol{\phi}(x_z) = ax_z^2$, where the trajectories converge to.

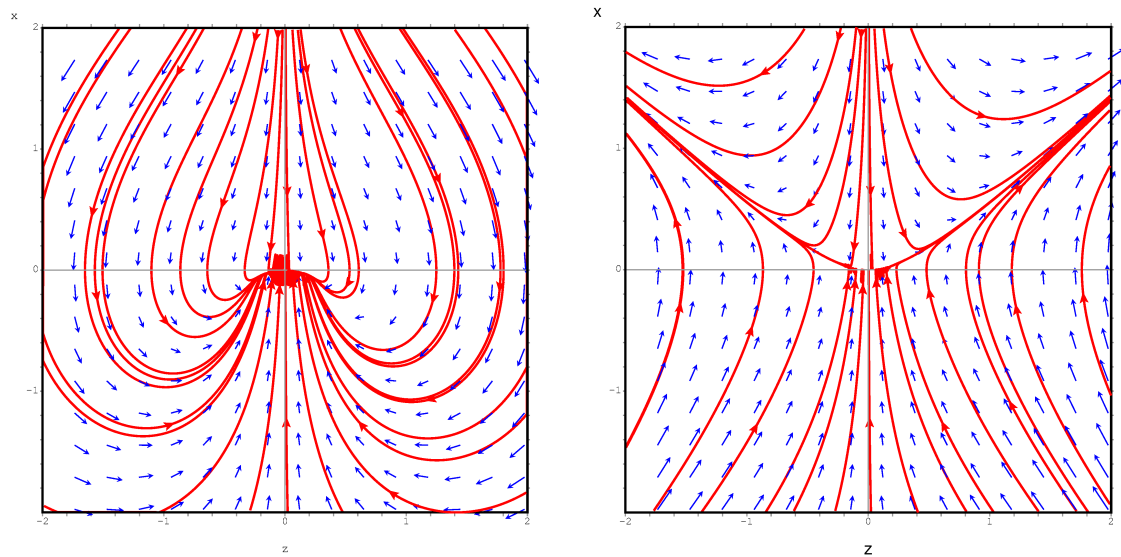


Figure 4.4: Phase plane of the dynamical system (4.16) for $a = -1$ (left), and $a = 1$ (right). The zero manifold $h(x_z)$ around the origin can be appreciated according to the approximation ϕ in (4.17).

4.2 Non-local phenomena

Until this point the discussion was about the qualitative behavior of the flow close to equilibrium points. This is also referred to local behavior. Unless this is the basis for most of the practical situations, basic knowledge about nonlocal phenomena is often necessary for the prediction, the design, and the control of nonlinear systems. For this purpose, in the present section, the basics of nonlocal analysis of dynamical systems are provided.

4.2.1 Index theory

A powerful tool for the analysis of nonlocal phenomena of nonlinear flows in two-dimensional systems consists in analyzing the changes in the direction of the vector field, while passing along curves, or over some region. In this section such an approach is presented in what concerns the so-called index theory for nonlinear systems. The presentation is restricted to the two-dimensional case. Nevertheless, some of the results can be extended to higher-dimensional systems.

In what follows consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2), & x_1(0) &= x_{10} \\ \dot{x}_2 &= f_2(x_1, x_2), & x_2(0) &= x_{20}\end{aligned}\tag{4.18}$$

For this kind of systems the following definition is of interest.

Definition 4.3

The index of a simple closed curve C (i.e. which does not intersect itself) that does not pass through any equilibrium point, indicates how many times the vector field rotates while passing counterclockwise along this curve, i.e.

$$I_C = \frac{1}{2\pi} \oint_C d\left(\arctan\left(\frac{f_2}{f_1}\right)\right) = \frac{1}{2\pi} \oint_C \frac{(df_2)f_1 - f_2(df_1)}{f_1^2 + f_2^2}\tag{4.19}$$

Given that at the starting and return point $\mathbf{x}_*$, the angle of the vector field at $\mathbf{x}_*$ is the same, the index I_C of any simple closed curve which does not contain equilibrium points, is an integer (the integral is necessarily a multiple of 2π). The idea behind studying the index is just the same as expressed by the Gauss-Green theorem, stating that the change within a closed region is given by the flux through the boundary of that region. This concept is well-used in electrostatics, and in chemical engineering.

Unless the index I_C of a curve C can in principle be calculated given the functions f_1 , and f_2 , it is more practical to analyze it graphically. To see how this works, consider the examples presented in Figure 4.5 illustrating two cases of vector fields with a curve C of index $I_C = 1$. To determine the index graphically, a simple trick [Str94] is to number the vectors counterclockwise, and copy them to a new coordinate chart, as illustrated in Figure 4.5.

From the definition and looking at Figure 4.5, the reader should be able to show that the index of a curve enclosing one isolated stable equilibrium point is equal to one. The same holds for a closed curve around an isolated repulsor node. This gives an idea that the index of a closed curve depends on the properties of the equilibrium points enclosed by it. This, in turn, gives rise to the definition of the index of an equilibrium point.

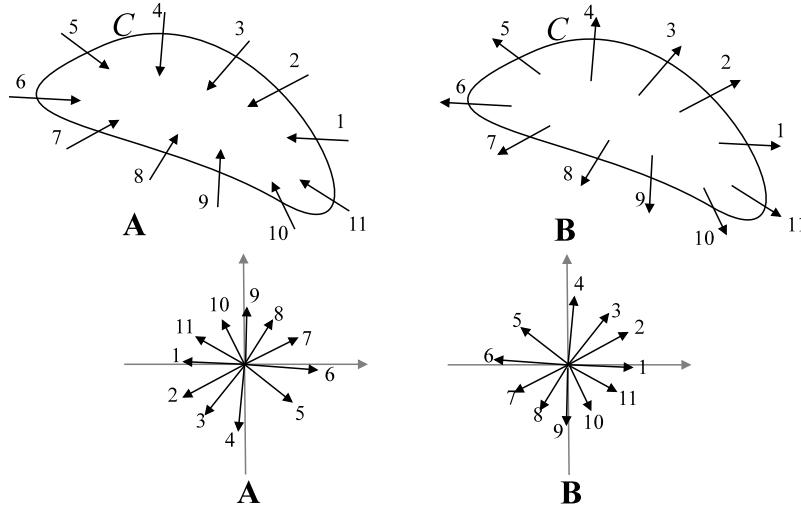


Figure 4.5: Illustration of two cases with closed curve C with index $I_C = 1$, and the associated vector direction change diagram.

Definition 4.4

The index $I[\mathbf{x}^*]$ of an isolated equilibrium point $\mathbf{x}^*$ is given by the index I_C of any simple closed curve C that encloses only this equilibrium point.

From this definition it should already be clear that the index of any closed curve which encircles the same isolated equilibrium point is the same. Actually, the curve can be shrunk or expanded without changing its index, as long as it does not touch other equilibrium points. Thus, the index is a strong topological measure for what happens qualitatively within a certain region, and gives a lot of information about non-local behavior.

You can think about what the index of a saddle point, or a stable (or unstable) spiral is, and verify yourself that a saddle node has index $I = -1$, and a stable (or unstable) spiral index $I = 1$. Clearly, if no equilibrium point is enclosed by a curve C , the index $I_C = 0$.

The next question to be addressed is about the index of a curve which encloses more than one isolated equilibrium point. How is the index of such a curve related to the indexes of the encompassed equilibrium points? The answer is provided in the next theorem.

Theorem 4.4

The index I_C of a simple closed curve C is equal to the sum of the indexes of the equilibrium points enclosed by it, i.e.

$$I_C = \sum_{i=1}^k I[\mathbf{x}_i^*] \quad (4.20)$$

where k is the number of equilibrium points $\mathbf{x}_i^*$ encompassed by C .

The idea behind the proof of this theorem is simple. As discussed above, the index of an equilibrium point is identical to the index of any simple closed curve encircling it. The index of a closed curve is invariant under stretching and shrinking, as long as no other equilibrium point is touched. To understand the idea, consider the case illustrated in Figure 4.6 for $k = 3$. The curve C is shrunk to

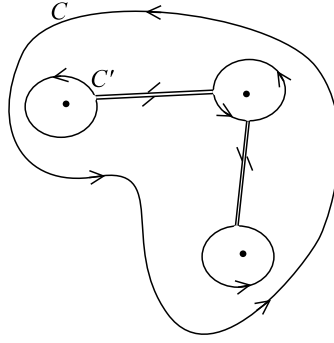


Figure 4.6: Illustration of the idea behind the proof of Theorem 4.4 for $k = 3$.

the curve C' with circles about the equilibrium points, joined by bridges which are passed once in both directions, thus mutually cancelling their effects in the integral (4.19). Accordingly, the index is given exactly by the sum of the three indexes of the equilibrium points. This idea can be applied to arbitrary numbers k , thus concluding the general relation (4.20).

4.2.2 Isolated closed orbits

A typical nonlinear non-local phenomenon is the existence of isolated periodic (i.e., closed) orbits, called **limit cycles**. To distinguish a limit cycle from a linear center, note that a linear center is surrounded by a continuum family of periodic orbits, while a limit cycle is isolated, i.e. there exists a neighborhood of the limit cycle which does not contain any other closed orbit. To highlight these properties the following definition is given.

Definition 4.5

A solution $\mathbf{x}(t) = \boldsymbol{\gamma}(t)$ of the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is called a **limit cycle** if it is an isolated periodic solution, i.e.

- (i) there exists a number $T > 0$ so that $\boldsymbol{\gamma}(t + T) = \boldsymbol{\gamma}(t)$ for all $t \geq 0$, and
- (ii) there exists a neighborhood $\mathcal{N}_{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ so that there are no other periodic solutions in $\mathcal{N}_{\boldsymbol{\gamma}}$.

There are many different everyday phenomena which are related with limit cycles, just as seasons, hart rythms, sleep-awake cycles, chemical self-sustained oscillations, electrical and electronic oscillators, etc. (for other examples see e.g. [Str94]).

Limit cycles can be stable or attracting (asymptotically stable), or unstable or repelling, in the sense of stability of sets discussed in Section 2.3. They may even be nested, i.e. there may be limit cycles enclosing other limit cycles. Limit cycles may collapse between each other, giving rise to non-local bifurcations, or simply break down. These issues will be analyzed later, and first a simple example is presented.

Example: Consider the following dynamics

$$\begin{aligned} \dot{x}_1 &= p x_1 - x_2 - p x_1 \sqrt{x_1^2 + x_2^2}, & x_1(0) &= x_{10} \\ \dot{x}_2 &= x_1 + p x_2 - p x_2 \sqrt{x_1^2 + x_2^2}, & x_2(0) &= x_{20} \end{aligned} \tag{4.21}$$

with $p > 0$. This dynamics becomes in polar coordinates

$$\begin{aligned} \dot{r} &= pr(1-r), & r(0) &= r_0 \\ \dot{\theta} &= 1, & \theta(0) &= \theta_0. \end{aligned} \quad (4.22)$$

Given that the analytic solution for θ is given by

$$\theta(t) = \text{mod}(\theta_0 + t, 2\pi)$$

implying that the trajectories spin around the origin with constant velocity, the interesting part is the dynamics of the radius. To analyze what happens to the radius r , consider the plot of the change over time (the right hand side), over the radius, qualitatively shown in Figure 4.7. Clearly, the equilibrium

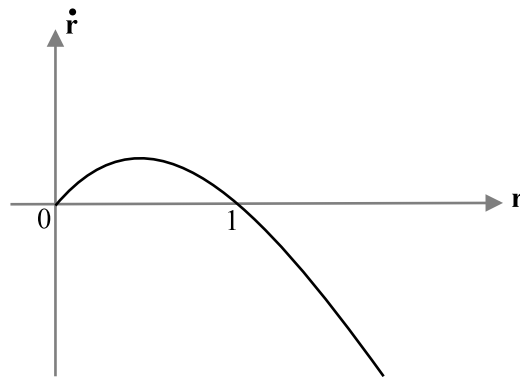


Figure 4.7: Illustration of the graph of $\dot{r}$ (the right-hand-side of (4.22)) versus r .

values for the radius are given by $r = 0$ and $r = 1$, and looking at the local slope at these equilibria (see Figure 4.7 reveals that $r = 0$ is unstable, while $r = 1$ is an attractor. Looking at the non-local behavior of $\dot{r}$ far from equilibrium, it becomes clear that for $r_0 < 1$ $\dot{r} > 0$ and thus the solution $r(t)$ will grow up until it reaches $r = 1$, while for $r_0 > 1$ $\dot{r} < 0$ and so $r(t)$ will decrease and reach $r = 1$ asymptotically. Thus, $r = 1$ is the unique attractor value for the radius. Given that the phase θ monotonically increases, the trajectory will never reach an equilibrium but will converge towards a closed orbit with radius $r = 1$. This behavior is shown in Figure 4.8 $\diamond$

Note that any limit cycle C is a simple closed curve, and thus has an associated index I_C . Recalling from the preceding section on index theory, how to determine the index of a closed curve, it is easy to see that the index of a limit cycle (stable or unstable) is $I_C = 1$. Given that a closed curve which does not enclose any equilibrium point has index 0, it follows directly that **a limit cycle necessarily encloses at least one equilibrium point**. Furthermore, given that the index of the closed curve is equal to the sum of indexes of the enclosed equilibrium points, it follows that these necessarily sum up to 1. This result is very util when analyzing the possibility of limit cycles in two-dimensional systems. See e.g. [SAL12] for an application example in the bioengineering branch.

To conclude this section, have a look at the Taylor linearization about the periodic limit cycle parameterized by time $\gamma(t)$, given by

$$\dot{\tilde{x}}(t) = J[\gamma(t)]\tilde{x}(t), \quad \tilde{x}(t_0) = \tilde{x}_0, \quad J[\gamma(t+T)] = J[\gamma(t)] \quad (4.23)$$

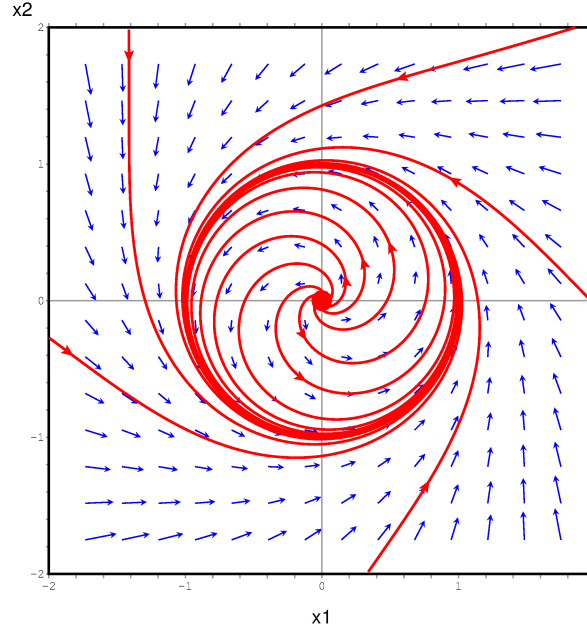


Figure 4.8: Phase plane of the nonlinear system (4.21) (or equivalently (4.22)) for $p = 1$, showing how the trajectories converge to a stable limit cycle with radius $r = 1$.

where $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \boldsymbol{\gamma}(t)$, and J is the time-varying periodic Jacobian evaluated along the limit cycle $\boldsymbol{\gamma}(t)$. According to Floquet theory (see Section 3.2), there exist a T periodic matrix $\psi(t)$ and a constant matrix B such that the associated fundamental matrix $P(t)$ (3.38) is given by (see Theorem 3.2.1)

$$P(t) = \psi(t)e^{Bt}.$$

The solution of the linearization is then given by

$$\tilde{\mathbf{x}}(t) = P(t)P^{-1}(t_0)\tilde{\mathbf{x}}_0 = \psi(t)e^{tB}P^{-1}(t_0)\tilde{\mathbf{x}}_0$$

The eigenvalues of B are the Floquet exponents, and their real parts are the Lyapunov exponents. Accordingly, if the Lyapunov exponents are all negative, the limit cycle $\boldsymbol{\gamma}$ is asymptotically stable (locally, i.e. in some tube around it), and if the Lyapunov exponents are positive, the limit cycle is unstable.

The Floquet exponents normally have to be calculated numerically. Herefore, a functional approximation of the limit cycle $\boldsymbol{\gamma}(t)$ and the period T have to be obtained first¹. Next, the system dynamics has to be linearized along the curve $\boldsymbol{\gamma}(t)$ (or an approximation of it), and the matrix differential equation

$$\dot{P}(t) = J[\boldsymbol{\gamma}(t)]P(t), \quad P(0) = I, \quad t \in [0, T]$$

has to be solved over a period. The solution matrix $P(T)$, evaluated at the period T , has the Floquet multipliers as eigenvalues, and the Floquet exponents are the eigenvalues of the associated matrix

$$B = \frac{1}{T} \ln[P(T)]. \quad (4.24)$$

This procedure is illustrated next for the dynamics (4.21).

¹This can be done through a Poincaré map.

Example: Consider the dynamics (4.21). The limit cycle coincides with the unit circle and is passed through with radial velocity $\dot{\theta} = 1$, i.e. with period $T = 2\pi$, and can be parameterized by sinus-cosinus functions as e.g.

$$\gamma(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}.$$

or equivalently, according to (4.22), in polar coordinates

$$\lambda(t) = \begin{bmatrix} r^*(t) \\ \theta^*(t) \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix}$$

With $\xi_0 = \begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix}$ we have

$$e(t) := \xi(t) - \lambda(t) = \begin{bmatrix} r(t) - r^*(t) \\ \theta_0 + t - \theta^* \end{bmatrix}$$

with linearized dynamics

$$\dot{e}(t) \approx J[\lambda(t)]e(t) = \begin{bmatrix} -p & 0 \\ 0 & 0 \end{bmatrix} e(t),$$

and the associated differential equation for the fundamental matrix $P(t) = \{p_{ij}(t)\}_{ij}$

$$\dot{P}(t) = J[\lambda(t)]P(t) = \begin{bmatrix} -pp_{11} & -pp_{12} \\ 0 & 0 \end{bmatrix}$$

with analytic solution

$$P(t) = \begin{bmatrix} e^{-pt}p_{11}(0) & e^{-pt}p_{12}(0) \\ p_{21}(0) & p_{22}(0) \end{bmatrix} = \begin{bmatrix} e^{-pt} & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, the monodromy matrix is given by

$$M = P(2\pi) = \begin{bmatrix} e^{-2p\pi} & 0 \\ 0 & 1 \end{bmatrix}$$

with eigenvalues (the Floquet multipliers) $\mu_1 = e^{-2p\pi}, \mu_2 = 1 \in \bar{U} \subset \mathbb{C}$, the closed unit circle, implying stability of the limit cycle and asymptotic stability in the radial component. Furthermore

$$B = \frac{1}{2\pi} \log(M) = \begin{bmatrix} -p & 0 \\ 0 & 0 \end{bmatrix}$$

with the Lyapunov exponents (i.e. the real parts of the eigenvalues of B) $\sigma_1 = -p$ and $\sigma_2 = 0$.

Note that the zero Lyapunov exponent stems from the constant phase drift, implying a stable dynamics of the local deviation from the limit cycle solution starting with zero phase $\theta^*(0) = 0$, and that the attractivity of the radial component corresponds to the Lyapunov exponent $\sigma_1 = -p$.

Actually, in polar coordinates the jacobian is time-invariant, and thus the matrix B is the same as the jacobian J . Hence, the above steps should illustrate the connections and interpretation of the meaning of B in the Floquet theory, and give a clue on how to compute it in more difficult cases.

In the case that no analytic solution for $\gamma(t)$ is at hand, numerical simulations can be used to determine an approximation $\gamma_a(t)$, which can then be used to determine the Floquet exponents. From a practical point of view, knowledge about the Floquet exponents corresponds to knowledge about local characteristic times (recall the discussion in section 2.3 on exponential stability), and thus allow to establish more precise predictions.

4.2.3 Existence and non-existence of limit cycles

While in the preceding section fundamental properties of limit cycles were discussed, such as the necessity of an equilibrium point in the interior of the limit cycle, the fundamental question on when they exist or not, and how to detect them is addressed in this section.

To be short, in general it is very hard to analytically detect a limit cycle, or even conclude about its existence. For particular classes of systems, like Lienard systems, introduced in the first subsection, it is possible to formulate conditions under which limit cycles occur. For general systems, almost the only result ensuring the existence of a limit cycle is the Poincaré-Bendixson theorem, which will be discussed in the second subsection. Most of the known results on this subject, present conditions for preclusion of limit cycles, such as the Bendixson-Dulac theorem, and Lyapunov's direct method, which will be discussed afterwards.

4.2.3.1 Lienard systems

Consider the second order nonlinear differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \quad x(0) = x_0, \dot{x}(0) = v_0, \quad (4.25)$$

known as Lienard equation. A system governed by Lienard's equation is called a Lienard system. Lienard systems are frequently employed in modeling oscillation circuits.

In state variables, with $x_1 = x$, $x_2 = \dot{x}$, the system (4.25) is written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g(x_1) - f(x_1)x_2 \end{aligned} \quad (4.26)$$

Before going into the general properties of the functions f and g implying the existence of a limit cycle, let us consider a famous particular case, given by the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \quad \mu \geq 0. \quad (4.27)$$

This perfectly fits into the form (4.26) with $f(x) = \mu(x^2 - 1)$ and $g(x) = x$. The van der Pol equation was originally developed during the study of oscillations electric circuits with vacuum tubes [Pol27].

Consider the integral

$$F(x) = \int_0^x f(s)ds = \mu \left(\frac{1}{3}x^3 - x \right)$$

and the coordinate change

$$w = \dot{x} + F(x)$$

so that

$$\dot{w} = \ddot{x} + f(x)\dot{x} = -g(x) = -x$$

and the van der Pol equation is written equivalently as

$$\begin{aligned}\dot{x} &= w - F(x) \\ \dot{w} &= -x.\end{aligned}\tag{4.28}$$

Clearly, the nullcline of x (i.e. the curves in phase space where $\dot{x} = 0$) is the cubic curve

$$w = F(x) = \frac{\mu}{3}x(x^2 - 3)$$

with zeros at $x = \pm\sqrt{3}$, and $x = 0$, illustrated by the dotted line in Figure 4.9. For any point over (or

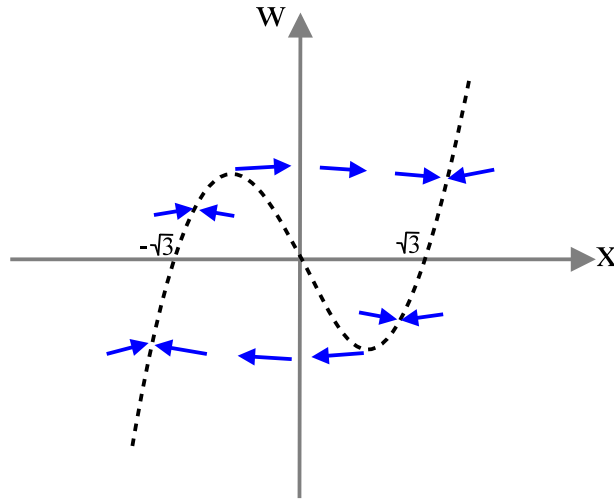


Figure 4.9: Illustration of the geometry of the vector field of the van der Pol oscillator in (x, w) -coordinates, according to (4.28).

under) this curve it holds that $\dot{x} > 0$ (or $\dot{x} < 0$), so that the flow is to the right (or to the left). The w -direction of the flow is upwards (or downwards) for $x < 0$ (or $x > 0$). This leads to the qualitative sketch of the vector field shown in Figure 4.9, and implying the existence of a limit cycle composed by a part sliding upwards the curve $w = F(x)$ on the left side, moving towards the right side, sliding down the curve $w = F(x)$, and jumping again towards the left side, and so on. This behavior can be observed in the phase portrait shown in Figure 4.10.

The parameter μ in (4.27) (or equivalently (4.28)) just defines the slope for the curve $w = F(x)$, and thus leads to a shorter, or longer sliding phase for smaller, or larger values of μ , respectively. Actually, for very large $\mu \gg 1$, the sliding phase becomes substantially longer, in comparison to the short jumping phase between the two sides of the curve $w = F(x)$, so that the system shows two different time scales: a slow one, corresponding to sliding, and a very fast one, corresponding to jumping. This behavior is illustrated in the time-response shown in Figure 4.11. Actually, one can show (see e.g. [Str94], and references therein) that the time-scale during sliding is $\mathcal{O}(\mu)$, while during jumping it is of order $\mathcal{O}(\mu^{-1})$, meaning that for $\mu = 10$ jumping is 100-times faster than sliding. This is an example of the so-called *relaxation oscillations* (see e.g. [Str94], and references therein).

Now, turning back to the general Lienard system (4.25), note that the line of reasoning that led us to conclude the existence of a limit cycle can be applied to any combination of functions f, g as long as these have some particular properties, which are stated in the next theorem.

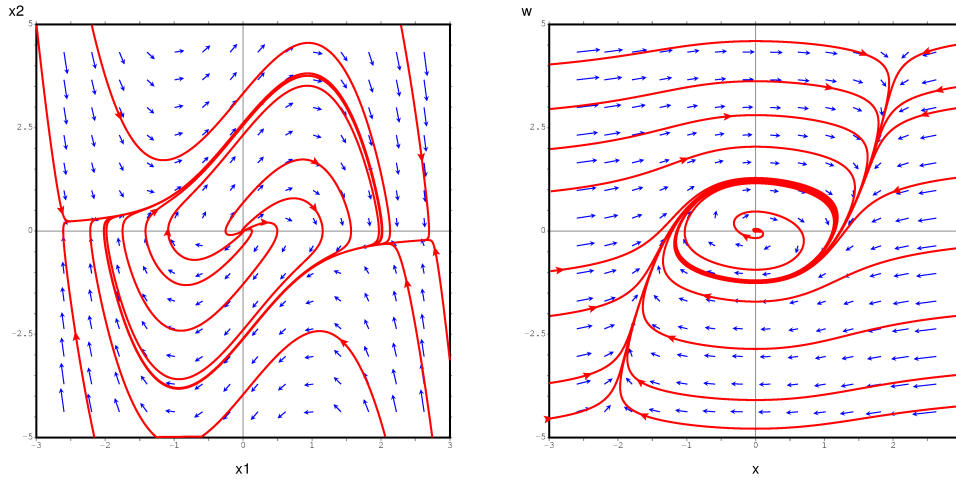


Figure 4.10: Illustration of the phase portrait of the van der Pol oscillator for $\mu = 2$ in state coordinates (4.26) (left) and equivalent coordinates (4.28) (right).

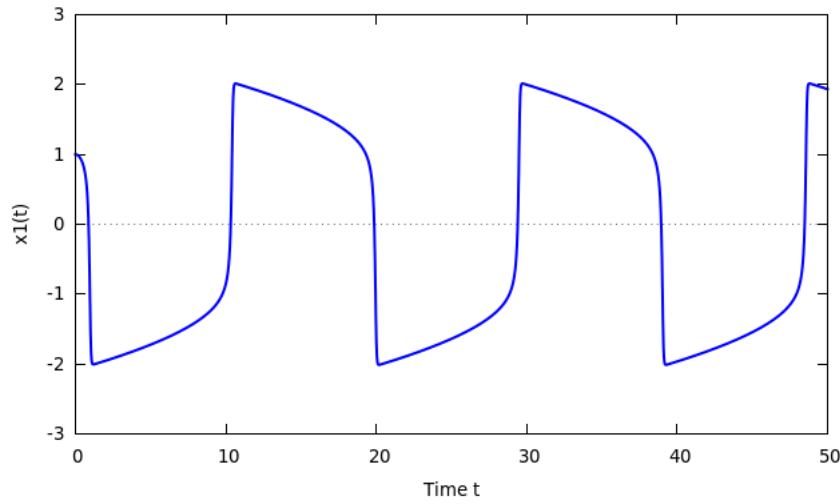


Figure 4.11: Time response of the van der Pol oscillator (4.27) for $\mu = 10$, showing the two-time scale behavior: a slow time scale during the sliding phase ($\mathcal{O}(\mu)$), and a fast one during the jumping phase ($\mathcal{O}(\mu^{-1})$).

Theorem 4.5

Consider the Lienard system (4.25), with $f, g \in \mathcal{C}^1$. If

- $g(-x) = -g(x)$ and $g(x) > 0$ for all $x > 0$ (i.e., g is contained in the first-third quadrant pair^a)
- $f(-x) = f(x)$, and $F(x) = \int_0^x f(s)ds$ has a unique positive zero at $x = a > 0$, $F(x) < 0$, $\forall 0 \leq x < a$, $F(x) > 0$, $\forall x > a$, and $F(x)$ is nondecreasing for $x > a$ with $F(x) \rightarrow \infty$ for $x \rightarrow \infty$

then there exists a unique asymptotically stable limit cycle enclosing the origin.

^aIn control literature, these kind of functions are related to passive static systems (see e.g. [SJK97]).

It is quite straightforward to show that in the case of the van der Pol oscillator these conditions are satisfied, and are exactly the ones underlying the above rationale for concluding the existence of a limit cycle.

4.2.3.2 The Poincaré-Bendixson Theorem

Considering a two-dimensional flow, the possibilities for a trajectory starting at a point $\mathbf{x}_0$ are quite restricted. Considering that there is no finite-escape, then asymptotically the trajectory may diverge, i.e. be unstable with norm tending to infinity, or may be bounded. In the last case, it is intuitively clear that a bounded trajectory can not spin around for all times arbitrarily because at no time instant it may intersect itself. This is so, because assuming such an intersection at time t_* in $\mathbf{x}_*$, there would be two distinct velocity vectors associated to $\mathbf{x}_*$, and thus two different trajectories starting at $\mathbf{x}_*$, which contradicts the existence and uniqueness theorem (see Figure 4.12). The only case where an apparent

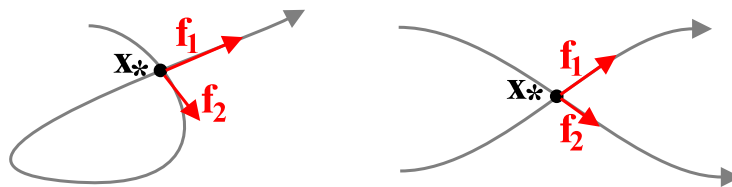


Figure 4.12: Illustration of the impossibility of intersections between trajectories in the plane.

intersection takes place is at equilibrium points. Nevertheless, this does not correspond actually to a real intersection, because the equilibrium is a limit for the trajectory for $t \rightarrow \infty$, if it is stable, or for $t \rightarrow -\infty$, if it is unstable.

The impossibility of intersections between trajectories (or of a trajectory with itself) implies that in the case of spiral movement, the spiral necessarily spins monotonically inward or outward, without changing direction, and thus a bounded spiral either converges towards an equilibrium point, or a periodic trajectory, i.e. a closed orbit.

The importance of the Poincaré-Bendixson Theorem consists in rigorously establishing that there are actually no other possibilities for bounded trajectories in the plane. The result was first established by the nonlinear dynamics pioneer Henry Poincaré, and later proven with alternative arguments by Ivar Bendixson. The theorem is stated next without proof in a convenient formulation. The interested reader can find alternative formulations, involving the notion of ω -limit sets, and the topologically interesting proof in the standard literature on dynamical systems (e.g. [Sas99; Wig03]).

Theorem 4.6: Poincaré-Bendixson

Let $\mathbf{x}(t)$ be a trajectory of (2.2) which evolves within a positively invariant compact set M . Then $\mathbf{x}(t)$ converges either to an equilibrium point, or to a closed orbit.

This result allows to conclude the existence of a limit cycle in a region of state space, but for its application it requires normally an additional concept, the one of a (positively) invariant set which has been introduced in Definition 2.7.

Invariant sets are sometimes called *trapping regions* for obvious reasons.

Having at hand the concept of compactness of a set M , the Poincaré-Bendixson theorem can be applied more easily, because the presence of an invariant set, ensures that all contained trajectories converge either to an equilibrium point or a periodic orbit. If there is only one equilibrium point in M , and this is a repulsor, like an unstable spiral, then there must necessarily be a stable limit cycle contained in M (see the right case illustrated in Figure 4.13).

Similar to this result is the statement, due to Andronov (see e.g. [Sas99]), that the only possibilities for bounded trajectories of nonlinear systems in the plane are given by or converge to

- attractive equilibria
- saddle connections (hetero- or homoclinic trajectories)
- limit cycles.

These three possibilities are illustrated in Figure 4.13.

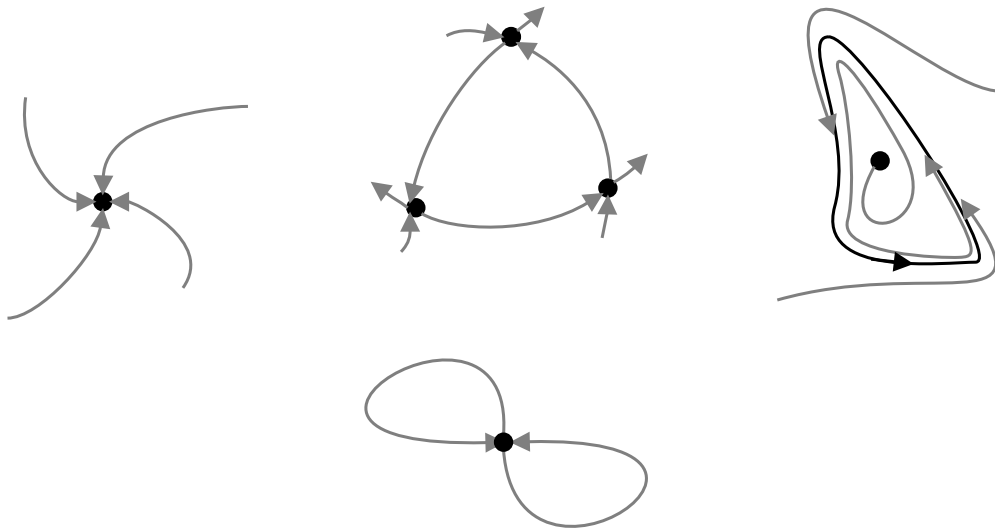


Figure 4.13: Three possibilities of bounded trajectories in the plane: convergence toward (i) an attractive equilibrium, (ii) a saddle connection, (iii) a limit cycle.

Example: The Glycolysis cycle

A simple model for the metabolic pathway for breaking down sugar for energy production, called glycolysis, is given by [Sel68]

$$\begin{aligned}\dot{x}_1 &= -x_1 + ax_2 + x_1^2 x_2 \\ \dot{x}_2 &= b - ax_2 - x_1^2 x_2\end{aligned}\tag{4.29}$$

Following the same line of reasoning as employed in [Str94], the local and non-local behavior will be discussed next. Setting the left-hand side of (4.29) to zero yields that there is exactly one equilibrium point

$$x_* = \begin{bmatrix} b \\ \frac{b}{b^2 + a} \end{bmatrix}.$$

The Jacobian matrix at this point is given by

$$J = \begin{bmatrix} -1 + 2\frac{b^2}{b^2 + a} & a + b^2 \\ -2\frac{b^2}{b^2 + a} & -(a + b^2) \end{bmatrix}$$

the trace is given by

$$\text{tr} = -\frac{1}{a + b^2}(b^4 + b^2(2a - 1) + (a + a^2)),$$

and the determinant by

$$\det = a + b^2 > 0$$

so that, according to (3.34), the limit between stable and unstable behavior of the equilibrium point is the curve

$$b^2 = \frac{1}{2}(1 - 2a \pm \sqrt{1 - 8a}).$$

The equilibrium point is locally stable for $\text{tr} < 0$ and unstable (i.e. repulsive) for $\text{tr} > 0$. This analysis corresponds to Hartman-Grobman and is valid in a sufficiently small neighborhood of the equilibrium point. Drawing the nullclines, i.e. the curves on which $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, one can see that they intersect only at $\mathbf{x} = \mathbf{x}_*$, and that the first quadrant is effectively separated into four different regions (see Figure 4.14, left part), where the vector field points

- I) towards the left (upward or downward)
- II) towards the top (left or right)
- III) towards the right (upward or downward)
- IV) towards the bottom (left or right)

Accordingly, trajectories will spiral around the equilibrium either inwards (stable) or outwards (unstable).

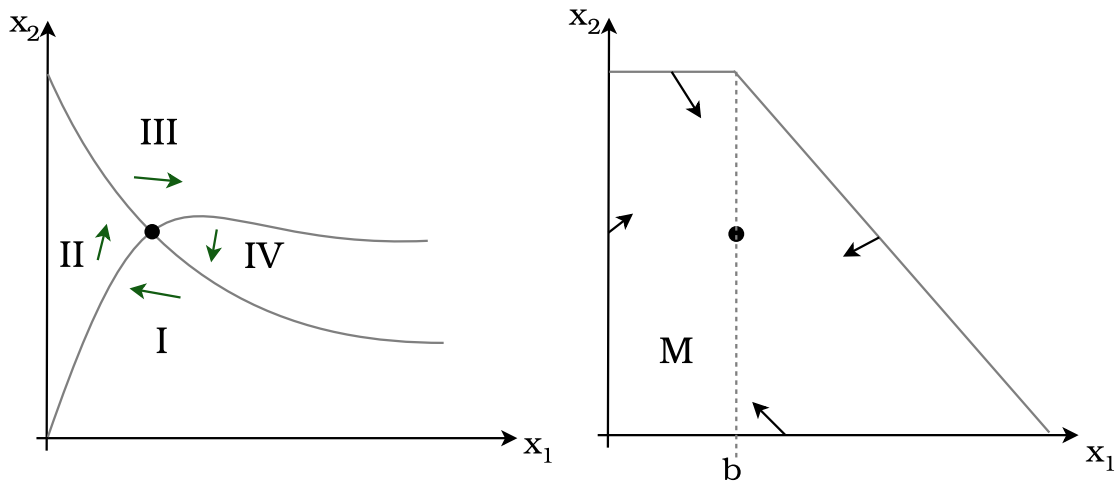


Figure 4.14: Schematical illustration of the flow of system (4.29) in the phase plane: (left) nullclines, and vector field around the equilibrium, (right) trapping region.

In order to find a trapping region in the (x_1, x_2) -plane, consider what happens at the boundaries of the first quadrant:

$$x_1 = 0 \Rightarrow \dot{x}_1 = ax_2 \geq 0, \quad \dot{x}_2 \begin{cases} \geq 0, & x_2 \leq \frac{b}{a} \\ < 0, & x_2 > \frac{b}{a} \end{cases}$$

$$x_2 = 0 \Rightarrow \dot{x}_1 = -x_1 \leq 0, \quad \dot{x}_2 = b > 0.$$

This shows that on the lower and left boundary of the first quadrant the vector field always points inwards the quadrant. On the other hand, introducing the variable

$$z = x_1 + x_2, \quad \dot{z} = -x_1 + b$$

one can see that the flow on the diagonal lines where $x_1 + x_2 = cst.$ is toward the left, as long as $x_1 > b$. So a trapping region can be constructed by moving this diagonal boundary sufficiently toward the right. To determine how far it has to be moved, note that the region will have to be completed by a *roof* with the only requirement that the flow is inward the trapping region, what is ensured if $\dot{x}_2 \leq 0$. Revising the dynamics (4.29), it can be seen that this requirement is fulfilled on the straight line $x_2 = b/a$ from $x_1 = 0$ to $x_1 = b$, given that over this interval

$$\dot{x}_2 = -\frac{b}{a}x_1^2 \leq 0.$$

This leads to the trapping region M illustrated in the right part of Figure 4.14. Together with the local analysis using the eigenvalue structure of the system discussed above, it follows from the Theorem of Poincaré-Bendixson, that in case of an (locally) unstable equilibrium point $\mathbf{x}_*$, there must exist a stable limit cycle enclosing it. To locate the limit cycle computer simulations have to be performed. In Figure 4.15 the two possible scenarios are shown: a stable spiral equilibrium (left: $a = 0.1, b = 0.2$), and an unstable spiral surrounded by a stable limit cycle (right: $a = 0.04, b = 0.5$).

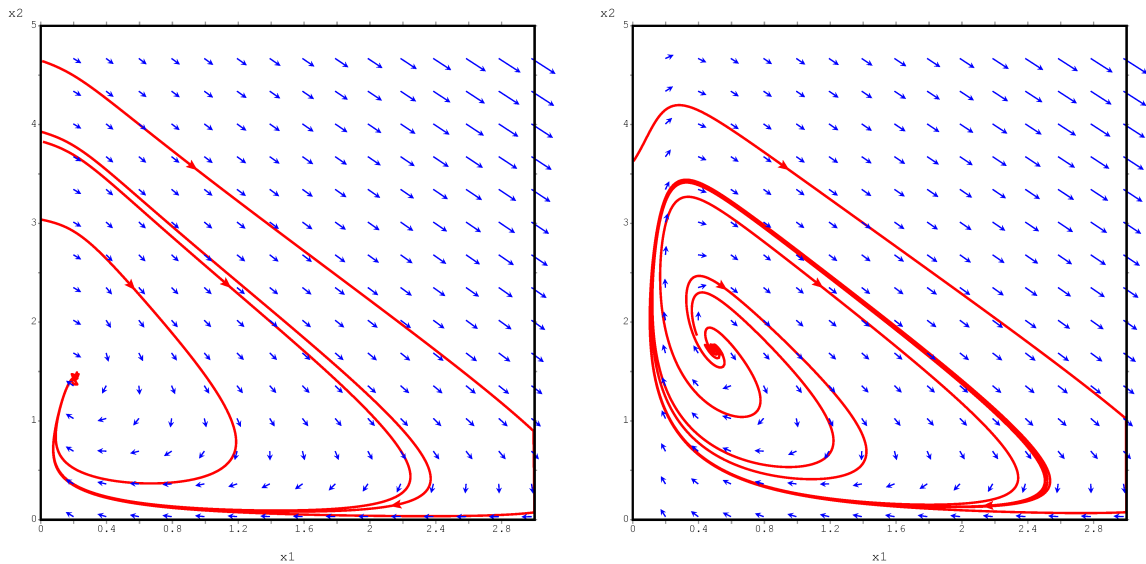


Figure 4.15: Simulation results for the glycolysis dynamics (4.29): (left) a stable spiral equilibrium with $a = 0.1, b = 0.2$, and (right) an unstable spiral surrounded by a stable limit cycle with $a = 0.04, b = 0.5$.

4.2.3.3 Bendixson-Dulac's non-existence criterion

In many applications it is of fundamental interest to establish the non-existence of limit cycles in a given region. A powerful tool for such a purpose is Bendixson's criterion and its refinement by Dulac, which is discussed next.

Consider a simply connected set $D \subset \mathbb{R}^2$, and consider a closed trajectory C lying entirely in D , as depicted in Figure 4.16. Let n denote the outward normal of C . Then it is clear that the scalar product of n and $\dot{x}$ at any point $x \in C$ is zero, i.e.

$$n \cdot \dot{x} = 0, \quad \forall x \in C.$$

Thus the same is true for the integral along C , i.e.

$$\oint_C \dot{x} \cdot n dl = 0$$

where dl is the element of arc length along C . Now, by Gauss' integral theorem, the preceding integral is equivalent to the integral of the divergence over the area A enclosed by the curve C , i.e.

$$\oint_C \dot{x} \cdot n dl = \iint_A \nabla \cdot \dot{x} da,$$

where da is an infinitesimal element of area. According to this relation the integral over the divergence has to be zero, what implies that the divergence necessarily has to be either zero or change its sign over the area A , which is contained in D . This means that if the divergence $\nabla \cdot \dot{x}$ is not zero and does not change sign over the whole domain D , no closed orbits can exist. This criterion for non-existence holds the name of Ivar Bendixson. It was later on generalized by Dulac, including a continuously differentiable function multiplying the vector field. Therefore, note that the same reasoning above applies for the vector field $\varphi(x)\dot{x}$, and no limit cycles, or any closed orbit can exist if $\nabla \cdot (\varphi(x)\dot{x})$ is not zero and does not change sign over the whole region D . This is summarized in the following theorem.

Theorem 4.7: Bendixson-Dulac

Let D be a simply connected subset of the plane. If there exists a $\mathcal{C}^1$ function $\varphi(x)$ such that $\nabla(\varphi\dot{x})$ is not zero anywhere in D and does not change its sign in D , then there are no closed trajectories contained in D .

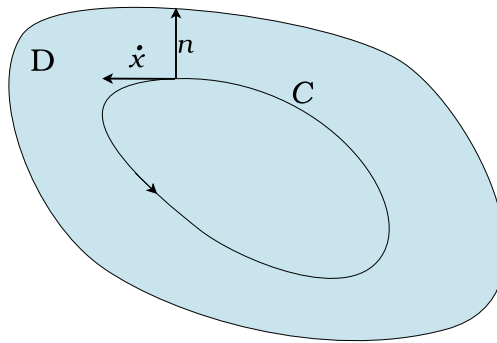


Figure 4.16: Geometric reasoning in the proof of the Theorem of Bendixson-Dulac.

4.3 Lyapunov's direct method

A very usefull way to establish the stability of an equilibrium point for a nonlinear dynamical systems consists in Lyapunov's direct method. Motivated by studies on energy dissipation in physical processes, in particular in astronomy, Aleksandr Mikhailovich Lyapunov, generalized these considerations to functions which are positive for any non-zero argument [Ful92]. In the sequel consider that the equilibrium point under consideration is the origin $\mathbf{x} = \mathbf{0}$. If other equilibria have to analyzed a linear coordinate shift $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$ can be employed to move the equilibrim to the origin in the coordinate $\tilde{\mathbf{x}}$.

To summarize the results of Lyapunov and generalizations of it some definitions are in order.

Definition 4.6

A continuous functional $V : \mathcal{D} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is called

- positive semi-definite if $\forall \mathbf{x} : V(\mathbf{x}) \geq 0$.
- positive definite if $\forall \mathbf{x} \neq \mathbf{0} : V(\mathbf{x}) > 0$ and $V(\mathbf{x}) = 0$ only for $\mathbf{x} = \mathbf{0}$.
- negative semi-definite if $\forall \mathbf{x} : V(\mathbf{x}) \leq 0$.
- negative definite if $\forall \mathbf{x} \neq \mathbf{0} : V(\mathbf{x}) < 0$ and $V(\mathbf{x}) = 0$ only for $\mathbf{x} = \mathbf{0}$.

Theorem 4.8

Let $V : \mathcal{D} \subset \mathbb{R}^n, V(\mathbf{x}) > 0$ be positive definite. If $\forall \mathbf{x} \in \mathcal{D} : \frac{dV}{dt}(\mathbf{x}) = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \dot{\mathbf{x}} \leq 0$, then $\mathbf{x} = \mathbf{0}$ is stable in the sense of Lyapunov.

Proof. Given that $V(\mathbf{x}) > 0$ and its continuity, there exists a function $W(\mathbf{x}) > 0$ such that

$$W(\mathbf{x}) \leq V(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{D}. \quad (4.30)$$

Let $\epsilon > 0$ and set

$$m := \min_{\|\mathbf{x}\|=\epsilon} W(\mathbf{x}) > 0. \quad (4.31)$$

Choose $\delta > 0$ such that

$$\max_{0 \leq \|\mathbf{x}\| \leq \delta} V(\mathbf{x}) \leq m.$$

Given that $m > 0, V(\mathbf{x}) > 0$ and the continuity of V such a positive δ always exists. It follows from the fact that V is non-increasing over time ($\dot{V}(\mathbf{x}) < 0$) that

$$\forall \mathbf{x}_0 : \|\mathbf{x}_0\| \leq \delta \quad \Rightarrow \quad V(\mathbf{x}(t; \mathbf{x}_0)) \leq m.$$

By (4.30) this implies that

$$W(\mathbf{x}(t; \mathbf{x}_0)) \leq m.$$

By virtue of the definition of m in equation (4.31) it follows that

$$\forall \mathbf{x}_0 : \|\mathbf{x}_0\| \leq \delta \Rightarrow \|\mathbf{x}(t; \mathbf{x}_0)\| \leq \epsilon$$

showing that the origin is stable in the sense of Lyapunov. $\square$

A function $V > 0$ that satisfies the conditions of Theorem 4.8 is called a *Lyapunov function*. Note that if $V > 0$ is continuously differentiable but it is not clear if $\frac{dV}{dt} \leq 0$ or the sign depends on some system parameters, then it is called a *Lyapunov function candidate*.

As can be seen from the proof, an essential part consists in that the sets

$$\Gamma_c = \{\mathbf{x} \in \mathbb{R}^n \mid V(\mathbf{x}) = c\} \quad (4.32)$$

defined by level curves of $V(\mathbf{x})$ are the boundaries of compact subsets $\mathcal{D}_c$ of the state space. In virtue of the non-increasing nature of V these sets are positively invariant. The geometric idea of the proof of Theorem 4.8 is quite beautiful and will be shortly discussed. See Figure 4.17 for an illustration. The conditions of the theorem ensure that for a given ϵ there exists a value $c > 0$ such that the set $\mathcal{D}_c$ with the boundary Γ_c defined in (4.32) is completely contained in the ϵ -neighborhood $\mathcal{N}_\epsilon$ of the origin, i.e. it holds that

$$\mathcal{D}_c \subseteq \mathcal{N}_\epsilon.$$

Choosing $\delta > 0$ such that the δ -neighborhood $\mathcal{N}_\delta$ is completely contained in $\mathcal{D}_c$ one obtains that

$$\mathcal{N}_\delta \subseteq \mathcal{D}_c \subseteq \mathcal{N}_\epsilon$$

with $\mathcal{D}_c$ being positively invariant. Thus it holds that for all $\mathbf{x}_0$ with $\|\mathbf{x}_0\| \leq \delta$, i.e. $\mathbf{x}_0 \in \mathcal{N}_\delta$ the solution $\mathbf{x}(t; \mathbf{x}_0)$ is contained in $\mathcal{D}_c \subset \mathcal{N}_\epsilon$, implying that $\|\mathbf{x}(t)\| \leq \epsilon$ for all $t \geq 0$.

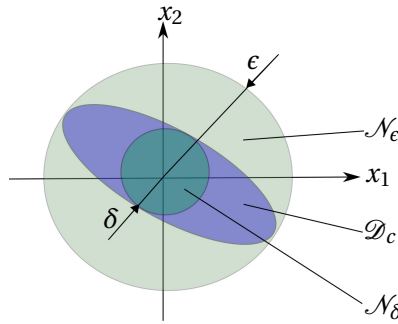


Figure 4.17: Geometrical idea behind the proof of Lyapunov's direct method in a two-dimensional state space.

The above only holds locally, unless $V(\mathbf{x})$ is strictly growing with $\|\mathbf{x}\|$. Thus the result is only local. The maximum compact set implied by the particular Lyapunov function can be explicitly determined. In the case that $\lim_{\|\mathbf{x}\| \rightarrow \infty} V(\mathbf{x}) = \infty$ the function is called *radially unbounded*. For a radially unbounded Lyapunov function the above result becomes global, i.e. it holds with $\mathcal{D} = \mathbb{R}^n$.

By evaluating explicitly the inequality $\frac{dV}{dt}(\mathbf{x}) = 0$ which holds over the set

$$\mathcal{X}_0 = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \frac{dV(\mathbf{x})}{dt} = 0 \right\} \quad (4.33)$$

one can apply the following result going back to Nikolay Nikolayevich Krasovskiy and Joseph Pierre LaSalle and is known as the *invariance theorem*.

Theorem 4.9

(Krasovskiy-LaSalle) Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a positively invariant compact set and $V \in \mathcal{C}^1(\mathcal{D} \rightarrow \mathbb{R})$ positive definite function with $\frac{dV}{dt}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathcal{D}$. Then the trajectories $\mathbf{x}(t)$ converge to the largest positively invariant set $\mathcal{M} \subseteq \mathcal{X}_0$ with $\mathcal{X}_0$ defined in (4.33).

If the conditions of this theorem are satisfied, an additional condition implies the asymptotic stability of the origin as stated next.

Theorem 4.10

If the conditions of Theorem 4.9 are satisfied and it holds that $\mathcal{M} = \{\mathbf{0}\}$, then the origin $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

A typical system where these results can be illustrated is given by the following Lienard oscillator

$$\ddot{x} + d\dot{x} + f(x) = 0 \quad (4.34)$$

with $d > 0$ and $f(x) > 0$ for $x > 0$, $f(x) = 0$ for $x = 0$ and $f(-x) = -f(x)$. The oscillator (4.34) can be written equivalently in state-space form with $x_1 = x$ and $x_2 = \dot{x}$ as

$$\dot{x}_1 = x_2 \quad (4.35a)$$

$$\dot{x}_2 = -f(x_1) - dx_2. \quad (4.35b)$$

Consider the following Lyapunov function candidate

$$V(\mathbf{x}) = \int_0^{x_1} f(\xi) d\xi + \frac{1}{2}x_2^2$$

motivated by the energy contained in the motion of $\mathbf{x}$ in form of potential and kinetic energy. The change in time of V is governed by

$$\begin{aligned} \frac{dV}{dt}(\mathbf{x}) &= f(x_1)\dot{x}_1 + x_2\dot{x}_2 \\ &= f(x_1)x_2 + x_2(-f(x_1) - dx_2) \\ &= -dx_2^2 \leq 0 \end{aligned}$$

implying stability of the origin $\mathbf{x} = \mathbf{0}$ in virtue of Theorem 4.8. From Theorem 4.9 it is additionally known that $\mathbf{x}$ converges into the set

$$\mathcal{X}_0 = \{\mathbf{x} \in \mathbb{R}^2 \mid x_2 = 0\},$$

and more specifically into the largest positively invariant subset of $\mathcal{M} \subseteq \mathcal{X}_0$. This set in turn contains only trajectories for which $x_2(t) = 0$ for all times, given that it is positively invariant. This means that $\dot{x}_2(t) = 0$ for all times. Substituting $x_2 = 0$, $\dot{x}_2 = 0$ into (4.35b) this means that $f(x_1(t)) = 0$ for all times, showing that

$$\mathcal{M} = \{\mathbf{0}\}$$

given that $f(x_1) = 0$ only for $x_1 = 0$. Corollary 4.10 implies that the origin $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

The asymptotic stability can also be concluded using Lyapunov's direct method if $\frac{dV}{dt}(\mathbf{x})$ is negative definite. This is stated in the next theorem.

Theorem 4.11

Let $V : \mathcal{D} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, $V > 0$. If $\forall \mathbf{x} \in \mathcal{D} : \frac{dV}{dt}(\mathbf{x}) < 0$, then $\mathbf{x} = \mathbf{0}$ is locally asymptotically stable in $\mathcal{D}$.

Proof. In virtue of Theorem 4.8 we have that $\mathbf{x} = \mathbf{0}$ is stable in the sense of Lyapunov. It thus remains to show that $\lim_{t \rightarrow \infty} V(\mathbf{x}) = 0$ to conclude, by taking into account the positive definiteness and the continuity of $V(\mathbf{x})$, that $\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = 0$.

Assume that V does not converge to zero. Then there exists a positive constant $c > 0$ such that $\lim_{t \rightarrow \infty} V(\mathbf{x}(t)) = c > 0$. Let

$$S = \{\mathbf{x} \in \mathcal{D} \mid V(\mathbf{x}) \leq c\}$$

By assumption, for $\mathbf{x}_0 \notin S$, i.e. $V(\mathbf{x}_0) > c$ it holds that $\forall t \geq 0 : \mathbf{x}(t) \notin S$. Let Γ_S be the boundary of the set S , i.e.

$$\Gamma_S = \{\mathbf{x} \in \mathcal{D} \mid V(\mathbf{x}) = c\}.$$

We have that $\dot{V}(\mathbf{x})|_{\mathbf{x} \in \Gamma_S} < 0$. Introduce

$$-\gamma := \max_{\mathbf{x} \in \Gamma_S} \dot{V}(\mathbf{x}) < 0.$$

Now, let $\mathbf{x}_0 \notin S$, i.e. $V(\mathbf{x}_0) > c$ and let $t > t_* := \frac{V(\mathbf{x}_0) - c}{\gamma}$. Observe that

$$\begin{aligned} V(\mathbf{x}(t; \mathbf{x}_0)) &= V(\mathbf{x}_0) + \int_0^t \dot{V}(\mathbf{x}(\tau; \mathbf{x}_0)) d\tau \\ &\leq V(\mathbf{x}_0) - \gamma t < V(\mathbf{x}_0) - \gamma t_* = c \end{aligned}$$

implying that $\forall t > t_*$ it holds that $\mathbf{x}(t; \mathbf{x}_0) \in S$. This contradicts the initial assumption that $\forall t \geq 0 : \mathbf{x}(t) \notin S$, and thus c cannot be positive and it must hold that $c = 0$. This, in turn, implies that $\lim_{t \rightarrow \infty} V(\mathbf{x}(t)) = 0$, and thus $\forall \mathbf{x}_0 \in \mathcal{D} : \lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = 0$. $\square$

At this place it is noteworthy that using Lyapunov functions one can establish a domain for which the equilibrium point is an attractor. This domain will always be included in the domain of attraction of the equilibrium point. Note that even though it is not possible to conclude if the domain of attraction established in this way is the complete domain of attraction or only a subset of it, unless the result is global.

As discussed above, in many cases it is not sufficient to conclude only the asymptotic stability and it becomes important to have a quantitative value for the convergence speed towards an equilibrium. This can be established if the equilibrium is exponentially stable (see Definition 2.4). Exponential stability can be concluded using Lyapunov functions if some additional properties are given. These are stated in the next theorem.

Theorem 4.12

Let $V : \mathcal{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}$, $V > 0$ be a positive definite functional. If there exist constants $\alpha, \beta, \gamma > 0$ so that

$$(i) \quad \alpha \|\mathbf{x}\|^2 \leq V(\mathbf{x}) \leq \beta \|\mathbf{x}\|^2 \quad (4.36a)$$

$$(ii) \quad \frac{dV}{dt}(\mathbf{x}) \leq -\gamma V(\mathbf{x}) \quad (4.36b)$$

then $\mathbf{x} = \mathbf{0}$ is exponentially stable and (2.16) holds with $a = \sqrt{\beta/\alpha}$ and $\lambda = \gamma/2$.

Proof. In virtue of (4.36b) it holds that

$$V(\mathbf{x}(t)) \leq V(\mathbf{x}_0) e^{-\gamma t}.$$

From (4.36a) this implies that

$$\|\mathbf{x}(t)\|^2 \leq \frac{1}{\alpha} V(\mathbf{x}(t)) \leq \frac{1}{\alpha} V(\mathbf{x}_0) e^{-\gamma t} \leq \frac{\beta}{\alpha} e^{-\gamma t} \|\mathbf{x}_0\|^2$$

and finally

$$\|\mathbf{x}(t)\| \leq \sqrt{\frac{\beta}{\alpha}} \|\mathbf{x}_0\| e^{-\frac{\gamma}{2} t}.$$

Exponential stability as defined in (2.16) follows with a and λ stated above. $\square$

Note that, in particular, the fact that the value of $V(\mathbf{x})$ monotonically decreases over time rules out the possibility of closed trajectories, given that they could only exist on level curves of V . Thus the use of Lyapunov functions is also an effective means for the preclusion of limit cycles.

Finally, it is possible to show that the existence of a Lyapunov function is intrinsically related to the stability properties of the equilibrium point as stated in the next theorem for the case of an exponentially stable equilibrium origin.

Theorem 4.13

Let $\mathbf{x} = \mathbf{0}$ be exponentially stable in $\mathcal{D} \subseteq \mathbb{R}^n$. Then there exists a Lyapunov function $V : \mathcal{D} \rightarrow \mathbb{R}$, $V(\mathbf{x}) > 0$ and a constant $\gamma > 0$ such that (4.36b) holds true.

Proof. By assumption there are constant a, λ such that $\|\mathbf{x}(t)\| \leq a \|\mathbf{x}_0\| e^{-\lambda t}$. Consider the functional

$$V(\mathbf{x}(t)) = \int_0^\infty \|\mathbf{x}(t+\tau)\|^2 d\tau.$$

It holds that

$$V(\mathbf{x}(t)) = 0, \quad \Leftrightarrow \|\mathbf{x}(t+\tau)\| = 0, \quad \forall \tau \geq 0,$$

showing that V is positive definite. On the other hand, in virtue of the exponential stability of the origin it holds that

$$V(\mathbf{x}(t)) \leq \int_0^\infty a^2 \|\mathbf{x}(t)\|^2 e^{-2\lambda\tau} d\tau = \frac{a^2 \|\mathbf{x}(t)\|^2}{2\lambda}$$

showing that for finite $\|\mathbf{x}_0\|$ the function $V(\mathbf{x})$ is quadratically bounded from above by the norm of $\mathbf{x}(t)$. Consider the rate of change of V at time t evaluated at the point $\mathbf{x}(t)$ given by

$$\begin{aligned}\frac{dV}{dt}(\mathbf{x}(t)) &= \lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \left(\int_{\tau}^{\infty} \|\mathbf{x}(t+s)\|^2 ds - \int_0^{\infty} \|\mathbf{x}(t+s)\|^2 ds \right) \\ &= - \lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_0^{\tau} \|\mathbf{x}(t+s)\|^2 ds \\ &= -\|\mathbf{x}(t)\|^2 \\ &\leq -\frac{2\lambda}{a^2} V(\mathbf{x}(t))\end{aligned}$$

showing that inequality (4.36b) holds with $\gamma = \frac{2\lambda}{a^2}$. □

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Bifurcations of vector fields

In this chapter, some fundamental behavior of nonlinear systems subject to parameter variations is analyzed and the concepts of bifurcation and structural stability are introduced. It is discussed how the qualitative behavior of the solution trajectories may be affected by bifurcations, and the most common bifurcations are classified.

5.1 Bifurcations of SSs

In comparison with linear systems, nonlinear systems have a much richer dynamics in the sense that very different phenomena can be observed. An impressive example is the existence of limit cycles that has been discussed in the preceding sections. Typically, nonlinear systems can also have multiple equilibria and even multiple attractors, each one with its own domain of attraction. In this section we will discuss further nonlinear phenomena that can occur due to parameter variations in dynamical systems.

First of all some qualitative definitions are in order. Throughout the preceding discussion, qualitative characterizations were normally provided which hold up to topological equivalence, in the sense that there exists a diffeomorphism (or homeomorphism) mapping the vector field of one system onto the one of the other system. Systems, for which for any (small enough) parameter variation such a diffeomorphism (or homeomorphism) exists are called **structurally stable** [AP37]. Nevertheless, as will become clear during the proceeding discussions, many nonlinear systems present a remarkable sensitivity with respect to parameter variations, leading to **structural instability**, meaning that no such diffeomorphism (or homeomorphism) exists. Such a lack of equivalence implies qualitative differences between the two vector fields of the same system for two different sets of parameters, and are called **bifurcations**.

These concepts should become clear in the following discussions. To exemplify the concept, imagine a system which for a given parameter has one equilibrium and for another (close) one two equilibria. Then it is not possible to map the vector fields onto one another in a bijective way, simply, because the zero vector from the one-equilibrium case would need to have two different images, namely the two equilibrium points for the other case.

5.1.1 Transcritical bifurcation

The most simple bifurcation of vector fields is achieved for the following dynamics

$$\dot{x} = rx - x^2, \quad x(0) = x_0 \quad (5.1)$$

The SSs are given by $x = 0$ and $x = r$, and coincide for $r = 0$. To analyze the stability properties of these SSs, in Figure (5.1) the right-hand side of (5.1) is qualitatively depicted in dependence of the state x , together with the associated vector field.

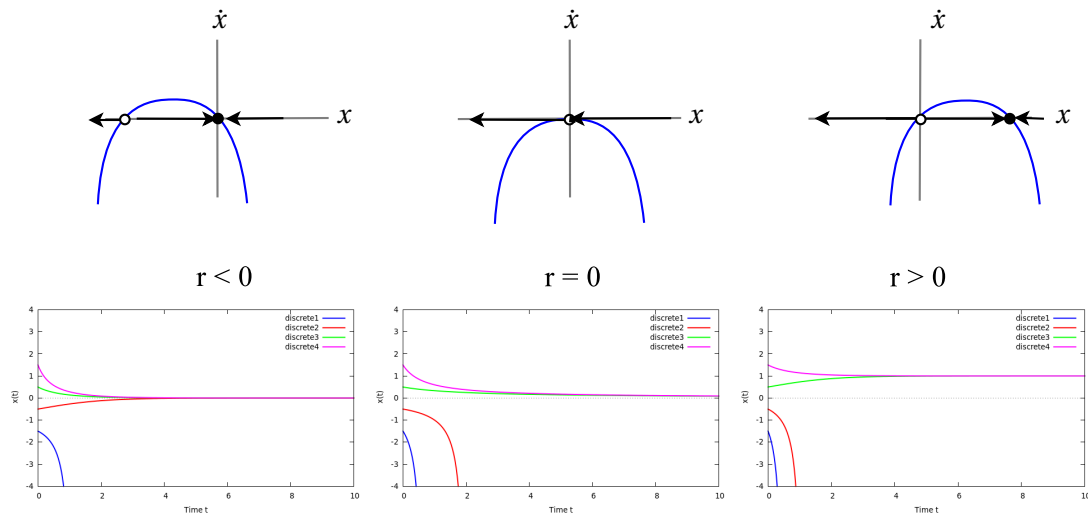


Figure 5.1: Illustration of the vector field and time responses for the transcritical bifurcation (5.1), and different values of the bifurcation parameter r : $r = -1$ (left), $r = 0$ (center), and $r = 1$ (right).

As it can be appreciated, there is a qualitative change in the behavior when the parameter passes through $r = 0$, given that both equilibria coincide for this value, and interchange stability. For $r < 0$, the origin is asymptotically stable and $x = r$ is unstable, while for $r > 0$, the origin is unstable and $x = r$ is asymptotically stable.

This can be illustrated in a compact way using the solution diagram presented in Figure 5.2. The continuous lines indicate locally asymptotically stable equilibria, and the discontinuous ones unstable equilibria.

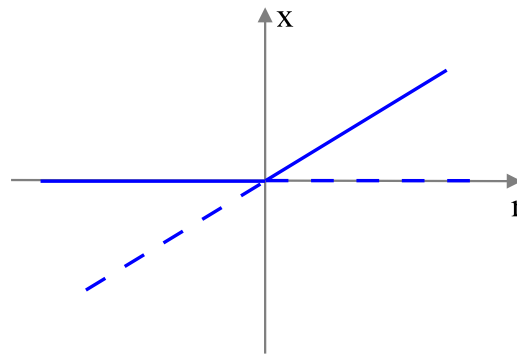


Figure 5.2: Solution diagram for the system (5.1) with transcritical bifurcation at $r = 0$.

Alternatively, the normalform of the transcritical bifurcation can be written with a sum instead of a subtraction, i.e.

$$\dot{x} = rx + x^2, \quad x(0) = x_0 \quad (5.2)$$

leading to an inverse solution diagram.

5.1.2 Saddle-Node (or Fold) bifurcation

Saddle-node bifurcations occur when an unstable and a stable node coincide, just as in the following system

$$\dot{x} = r - x^2, \quad x(0) = x_0 \quad (5.3)$$

The associated SS equation is given by

$$0 = r - x^2.$$

Clearly, for $r < 0$ there are no solutions, for $r = 0$ there is exactly one solution, and for $r > 0$ there are two solutions. This shows the sensitivity of SS multiplicity on the bifurcation parameter r . The local stability property follows from the sign of the slope at the equilibrium points, so that for $r > 0$ the positive equilibrium point at $x = +\sqrt{r}$ is asymptotically stable, and the negative one $x = -\sqrt{r}$ is unstable. The equilibrium at $r = 0$ is a saddle, because $\dot{r} < 0$ for any $x \neq 0$. A qualitative illustration of the right-hand side plotted against the x -axis, together with the associated vector field is presented in Figure 5.3.

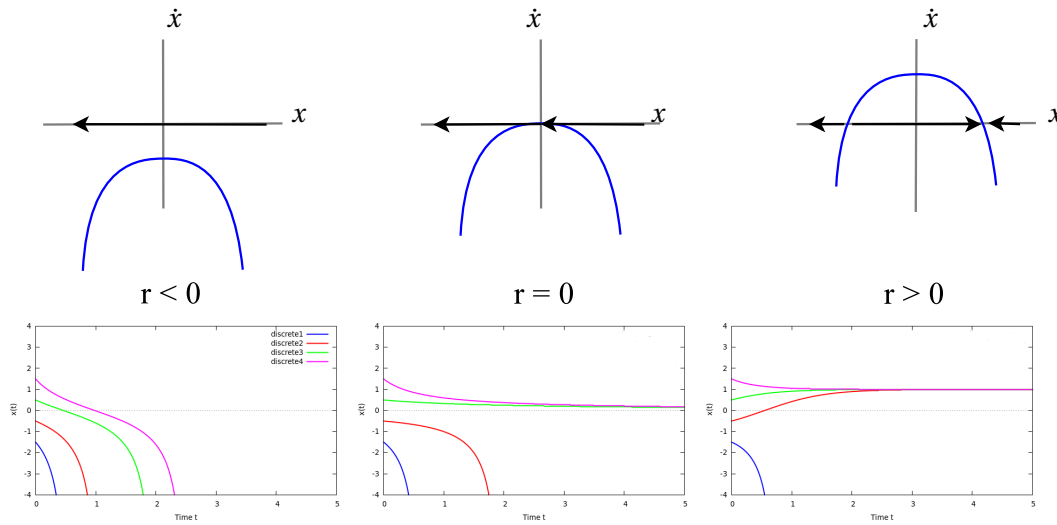


Figure 5.3: Above: Illustration of the vector field of (5.3) for three different cases: (left) $r < 0$, (middle) $r = 0$, and (right) $r > 0$. Below: Time responses associated to the three cases.

The associated solution diagram is depicted in Figure 5.4

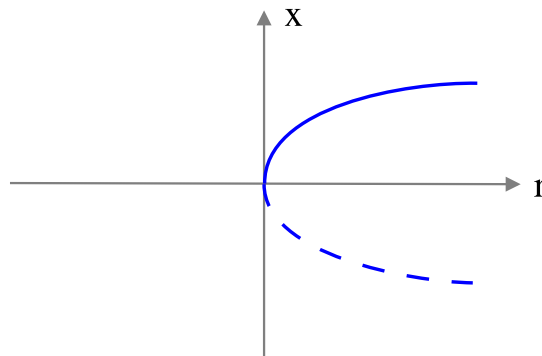


Figure 5.4: Solution diagram for the system (5.3) with saddle-node bifurcation at $r = 0$.

Again, the normalform of the saddle-node bifurcation can be written alternatively as

$$\dot{x} = r + x^2, \quad x(0) = x_0. \quad (5.4)$$

5.1.3 Pitchfork bifurcation

Consider the nonlinear system

$$\dot{x} = rx - x^3, \quad x(0) = x_0 \quad (5.5)$$

The equilibria are solutions of the equation

$$0 = x(r - x^2)$$

so that $x_1 = 0$, $x_{2,3} = \pm\sqrt{r}$ are solutions. Clearly, for $r < 0$, $x_1 = 0$ is the unique solution, while for $r > 0$ there are three solutions. The linearization of the dynamics close to x_i , $i = 1, 2, 3$ shows that for $r < 0$ the SS $x_1 = 0$ is asymptotically stable (globally, since it is the unique SS of a scalar flow), and for $r > 0$ it is unstable, surrounded by the two asymptotically stable (symmetric) SSs x_2 and x_3 . For $r = 0$ all SSs coincide that the trajectories converging from outside (e.g., with $x_0 > \sqrt{r}$) are attracted to the origin, and thus the origin is asymptotically stable. This is verified by looking at the graph of the right-hand side of (5.5) against x , with the associated vector field depicted on the x -axis, as shown in Figure 5.5.

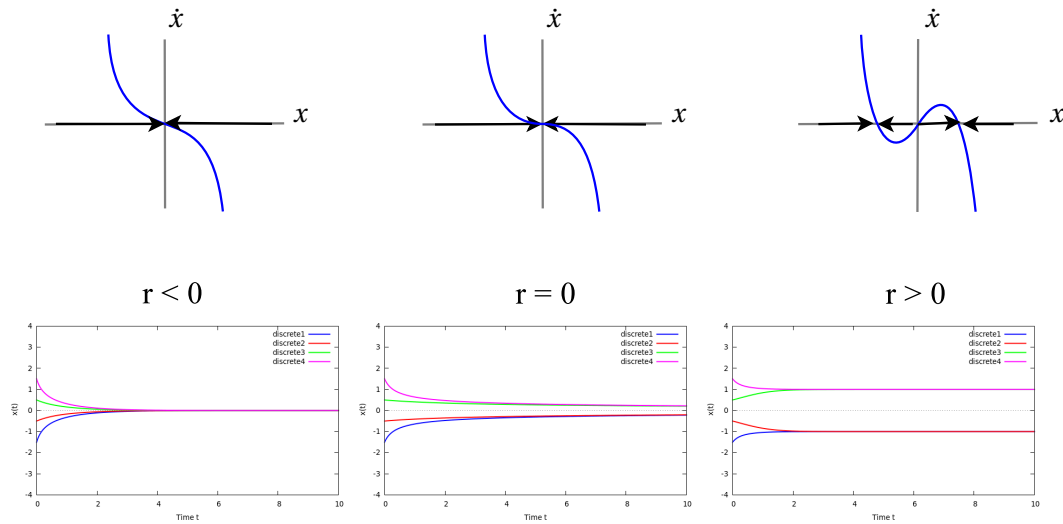


Figure 5.5: Illustration of the vector field and time responses for the pitchfork bifurcation (5.5), and different values of the bifurcation parameter r : $r = -1$ (left), $r = 0$ (center), and $r = 1$ (right).

The associated solution diagram is depicted in Figure 5.6

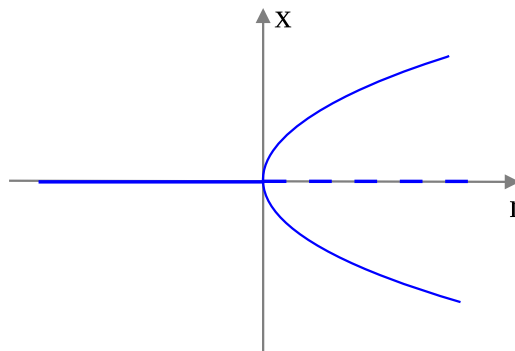


Figure 5.6: Solution diagram for the system (5.3) with saddle-node bifurcation at $r = 0$.

As in the previous cases, also the normalform of the pitchfork bifurcation can be written alternatively as

$$\dot{x} = rx + x^3, \quad x(0) = x_0. \quad (5.6)$$

This leads to the solution diagram shown in Figure 5.7

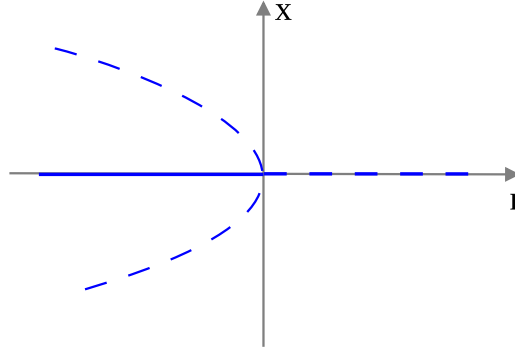


Figure 5.7: Solution diagram for the system (5.3) with saddle-node bifurcation at $r = 0$.

Comparing both solution diagrams a substantial difference turns out. With the negative sign in (5.5) the multiplicity is obtained for parameter values superior to the bifurcation (or critical) value $r = 0$. Thus this bifurcation is also called super-critical pitchfork bifurcation. With the positive sign in (5.6) the multiplicity is obtained for values below (sublimar to) the bifurcation (or critical) parameter value. In consequence, this bifurcation is called sub-critical pitchfork bifurcation. Besides this minor drastic point, the most important implication of the change in one sign resides in the fact that for the sub-critical case (5.6) the two additional steady-states are repulsors. Thinking about technical applications this means that in the local operation

the locally stable steady-state $x = 0$ for $r < 0$ not only turns unstable, but the trajectories diverge for positive r to infinity. In the super-critical case, instead of this behavior the steady-states appearing for positive r are attractors. In consequence, eventhough the globally asymptotically stable steady-state $x = 0$ still turns unstable, on both sides attractors appear retaining the solutions in a compact set around the origin and preventing them from diverging to infinity. Thus, the sub-critical pitchfork bifurcation is much more dangerous from an application-oriented point of view, because a system can turn completely unstable with a slight parameter change, while in the super-critical case (natural) mechanisms retain the trajectories in bounded sets.

5.1.4 Bifurcations of SSs in dimension $n > 1$

The preceding bifurcations are directly extended to the higher-dimensional case by adding independent (stable or unstable) directions. So the normal form for the transcritical, saddle-node, and pitchfork bifurcation in dimension 2 are given by

$$\begin{aligned} \dot{x}_1 &= rx_1 - x_1^2 \\ \dot{x}_2 &= \pm x_2 \end{aligned} \quad \text{transcritical bifurcation} \quad (5.7)$$

$$\begin{aligned} \dot{x}_1 &= r - x_1^2 \\ \dot{x}_2 &= \pm x_2 \end{aligned} \quad \text{saddle-node bifurcation} \quad (5.8)$$

$$\begin{aligned}\dot{x}_1 &= r x_1 - x_1^3 \\ \dot{x}_2 &= \pm x_2\end{aligned}\quad \text{pitchfork bifurcation}$$

(5.9)

The corresponding vector field for the case of asymptotically stable x_2 direction are illustrated in Figures 5.8-5.10.

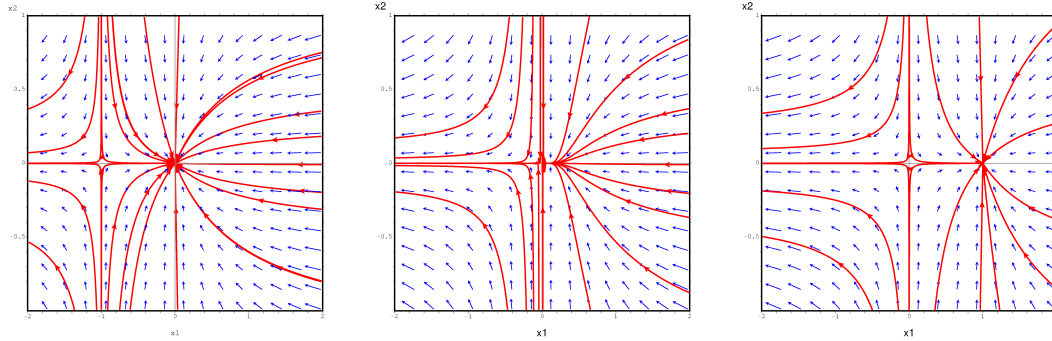


Figure 5.8: Phase plane of the two-dimensional transcritical bifurcation (5.7) with asymptotically stable x_2 -direction for $r = -1$ (left), $r = 0$ (center), and $r = 1$ (right).

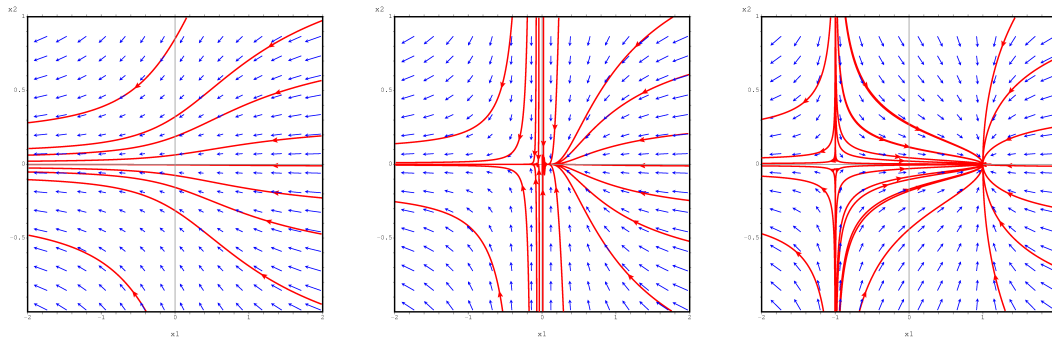


Figure 5.9: Phase plane of the two-dimensional saddle-node bifurcation (5.8) with asymptotically stable x_2 -direction for $r = -1$ (left), $r = 0$ (center), and $r = 1$ (right).

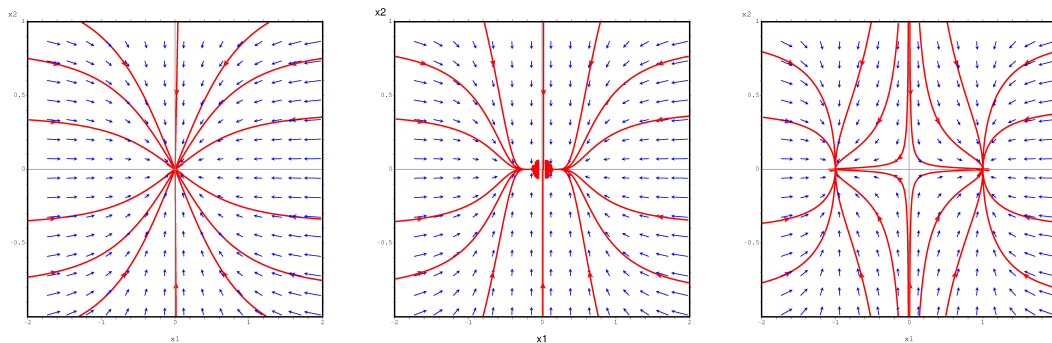


Figure 5.10: Phase plane of the two-dimensional pitchfork bifurcation (5.9) with asymptotically stable x_2 -direction for $r = -1$ (left), $r = 0$ (center), and $r = 1$ (right).

5.2 Bifurcations of trajectories

In the preceding section, bifurcations affected the number and stability properties of SSs. Many applications nevertheless present bifurcations that affect trajectories, and more particularly limit cycles. In this section the most important basic bifurcations of trajectories are briefly discussed.

5.2.1 The non-uniform oscillator (1D)

It should be noted that for flows on the line (1D) there are no periodic orbits possible, because at each point x there is a unique velocity vector $f(x)$ assigned, and thus a unique direction. Hence, it is impossible that any trajectory returns to its origin.

This is no longer the case when the state space is the circle (1D embedded in a 2D space) with state $\theta \in [0, 2\pi]$ being the angle of a point wandering on the circle (Figure 5.11).

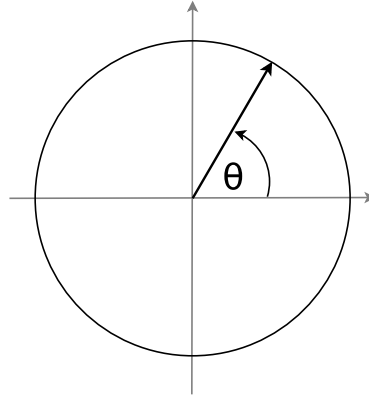


Figure 5.11: For flows on the circle, the angle is the one-dimensional (possibly periodic) state.

Probably the most typical example of dynamics on the circle is the **non-uniform oscillator**

$$\dot{\theta} = \omega - r \sin(\theta), \quad \theta(0) = \theta_0 \quad (5.10)$$

The most simplest case of dynamics of (5.10) is given for $r = 0$, and is called the **uniform oscillator**, given that the solution is a periodic oscillation with uniform velocity, i.e. with velocity that does not depend on the angle θ . The analytic solution in this case is given by

$$\theta(t) = \text{mod}(\theta_0 + \omega t, 2\pi) \quad (5.11)$$

and the time response is depicted in Figure 5.12.

A more interesting case is given for $r > 0$. When $0 < r < \omega$, there is still an oscillation but it is no more uniform, given that the velocity $\dot{\theta}$ now depends on the angle θ . When $r = \omega$, then a SS appears at $\theta = \pi/2$, attracting the flow from one side, and repelling it from the other (see Figure 5.13). When $r > \omega$, there are two SS solutions, one being a repulsor and the other an attractor.

This behavior remembers the saddle-node bifurcation (5.3). To analyze the saddle-node bifurcation hypothesis, consider the dynamics close to the bifurcation state $\theta = \pi/2$, introducing the variable

$$\tilde{\theta} = \theta - \frac{\pi}{2}$$

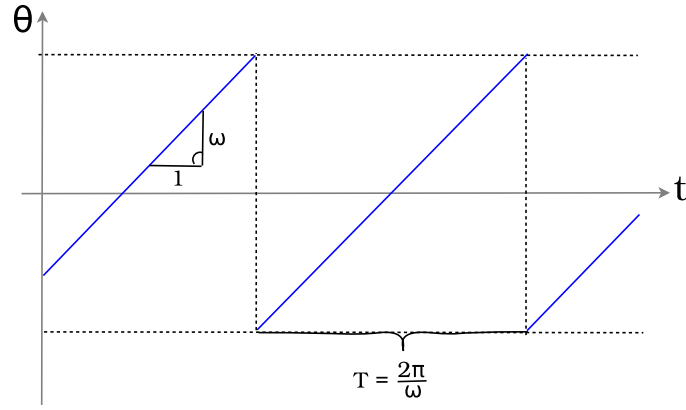


Figure 5.12: Time response of the uniform oscillator (5.10) with $r = 0$, and analytic solution (5.11).

with dynamics (using Taylor expansion)

$$\dot{\tilde{\theta}} = \omega - r \left[\sin\left(\frac{\pi}{2}\right) + \cos'\left(\frac{\pi}{2}\right)\tilde{\theta} - \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\tilde{\theta}^2 + \mathcal{O}(\tilde{\theta}^3) \right]$$

Neglecting terms of cubic and higher order one obtains

$$\dot{\tilde{\theta}} = (\omega - r) - \frac{r}{2}\tilde{\theta}^2.$$

Introducing the new time scale τ and variable r

$$\tau = \frac{r}{2}t, \quad p = \frac{2(\omega - r)}{r} \quad (5.12)$$

the preceding dynamics can be written as

$$\frac{d\tilde{\theta}}{d\tau} = p + \tilde{\theta}^2 \quad (5.13)$$

which is the normal form (5.4) of the saddle-node bifurcation.

Figure 5.13 shows the velocity over θ for these three cases. Figure 5.14 shows how the associated flow on the circle looks like.

It is quite noteworthy that for $r \lesssim \omega$ the SS is not present, but some strange behavior occurs when looking at the time response. This is illustrated in Figure 5.15 for $\omega = 1$ and $r = 0.95$. The trajectories approach the point $\theta = \pi/2$, significantly slow down, and seem to attain a constant value, while in reality they pass through a so called *bottleneck* or *ghost*, being a kind of artifact of the saddle-node bifurcation. The reason for this behavior resides in the fact that the angular velocity becomes very slow close to the point $\pi/2$, and nearly goes to zero. So, actually, the period T is almost given by the time required to pass through the bottleneck, and can be estimated for the general saddle-node bifurcation (5.3) as follows:

$$T = \int_0^T d\tau \approx \int_{-\infty}^{\infty} \frac{d\tau d\tilde{\theta}}{d\tilde{\theta}} = \int_{-\infty}^{\infty} \frac{d\tilde{\theta}}{p + \tilde{\theta}^2}$$

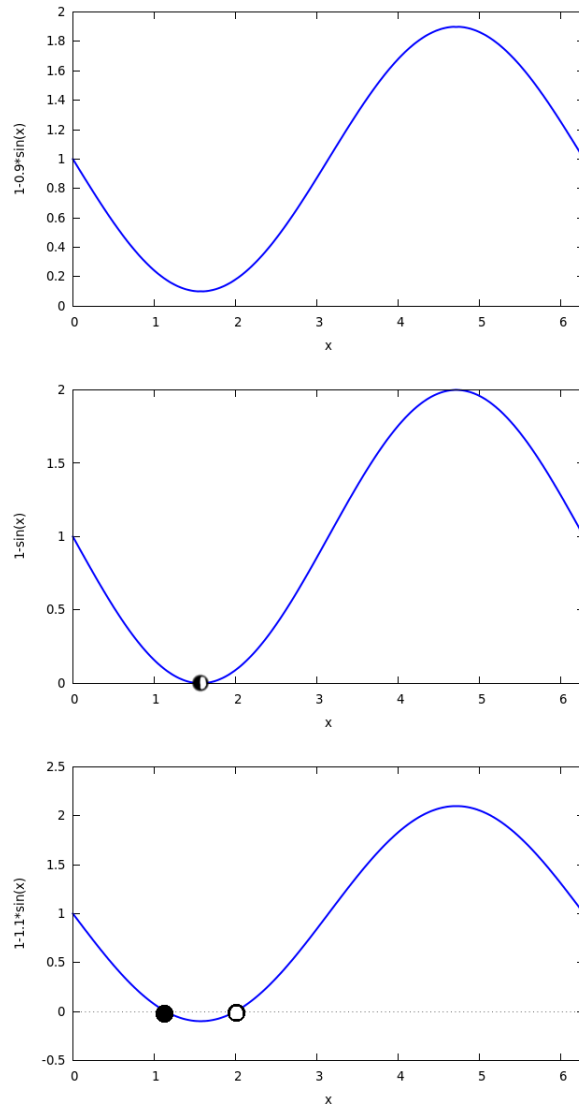


Figure 5.13: Velocity vs. angle for the non-uniform oscillator (5.10) for $\omega = 1$, $r = 0.9$ (top), $r = 1$ (center), and $r = 1.1$ (bottom).

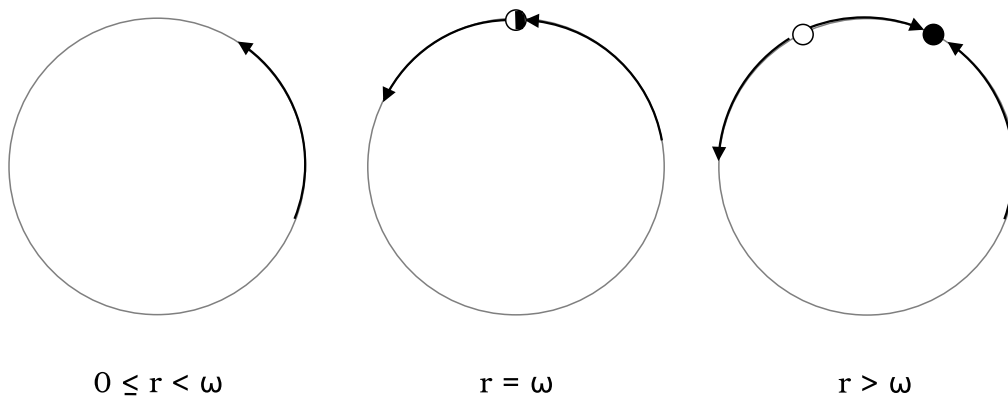


Figure 5.14: Bifurcations on the circle for the non-uniform oscillator. (left) periodic solution, (center) infinite-period (saddle-node) bifurcation, (right) two SSs on the circle, one attractor and one repulsor.

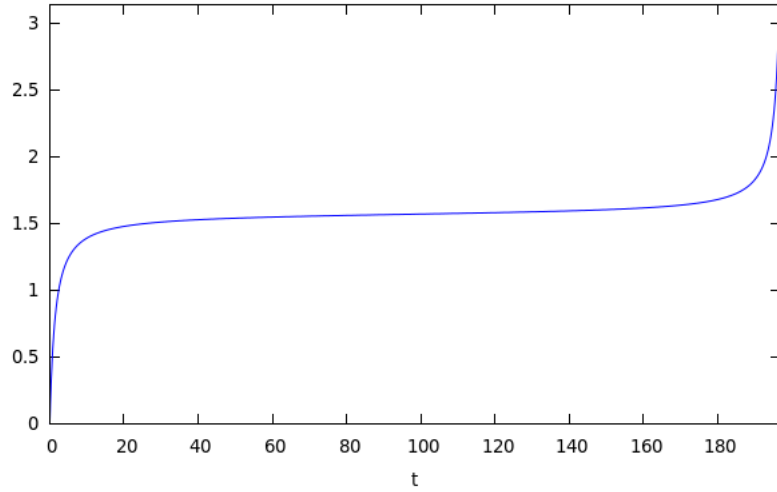


Figure 5.15: Numerical simulation over one period for the non-uniform oscillator (5.10) with $\omega = 1$, $r = 0.9995$, showing the passage through a bottleneck (or ghost).

Introducing the change of variable

$$\tilde{\theta} = \sqrt{p} \tan(x), \quad d\tilde{\theta} = \sqrt{p} \frac{1}{\cos^2(x)} dx$$

we obtain

$$T = \int_{-\pi/2}^{\pi/2} \frac{\sqrt{p} dx}{p \cos^2(x) \left(1 + \frac{\sin^2(x)}{\cos^2(x)}\right)} = \frac{1}{\sqrt{p}} \int_{-\pi/2}^{\pi/2} dx = \frac{\pi}{\sqrt{p}}$$

This shows that the time to pass through the bottleneck tends to infinity when $p \rightarrow 0$.

Applying the preceding relation to the non-uniform oscillator (5.10), it follows that for $r \lesssim \omega$ in terms of the time scale τ (5.12) the time to pass through the bottleneck is given approximately by

$$\tau_b \approx \frac{\pi}{\sqrt{p}} = \frac{\pi}{\frac{2(\omega - r)}{r}} = \sqrt{\frac{r}{2}} \frac{\pi}{\sqrt{\omega - r}} \quad (5.14)$$

or in original time scale (5.12)

$$T_b = \frac{2}{r} \frac{\sqrt{r} \pi}{\sqrt{2} \sqrt{\omega - r}} = \sqrt{\frac{2}{r}} \frac{\pi}{\sqrt{\omega - r}} \quad (5.15)$$

For the case example illustrated in Figure 5.15, this corresponds to $T_b \approx 198.74$, in accordance with the numerical simulation.

5.2.2 Andronov-Hopf bifurcation (2D)

Until now, all bifurcations involved the appearance or disappearance of SSs. A typical bifurcation where no SSs appear or disappear, but the behavior changes qualitatively (i.e. topologically) for small variations of some parameter is the Andronov-Hopf bifurcation. The typical change is illustrated in Figure 5.16, showing a decaying oscillation before the parameter change (top), and an increasing oscillation with constant (for a given parameter) limiting amplitude after the bifurcation. The typ-

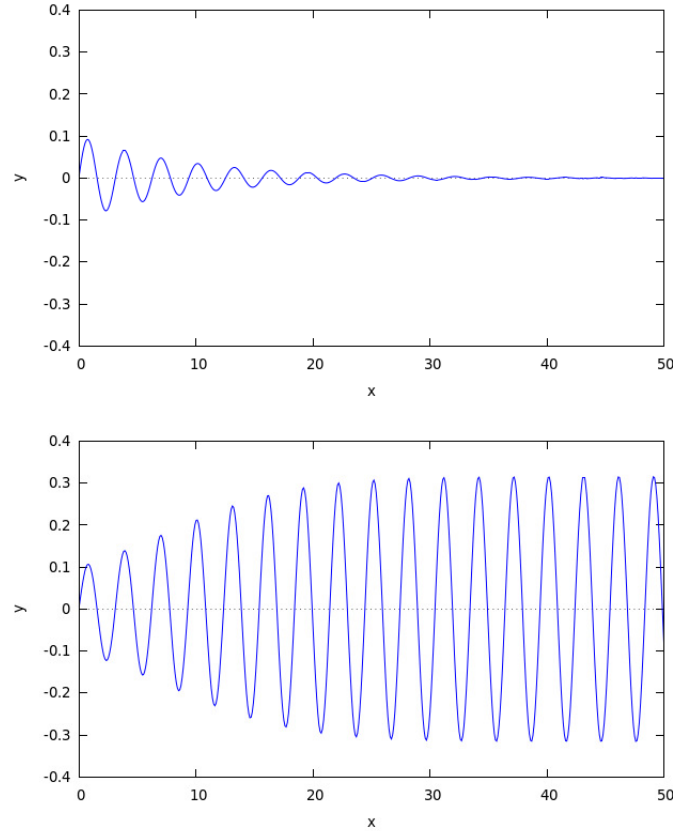


Figure 5.16: Typical behavior associated to a Hopf bifurcation: a decaying spiral before the parameter change (top), and an increasing oscillation with constant limiting amplitude after the parameter change (bottom).

ical dynamics of this behavior are described by the following set of differential equations in polar coordinates

$$\begin{aligned} \dot{r} &= pr - r^3 \\ \dot{\theta} &= \omega + br^2 \end{aligned} \tag{5.16}$$

which we analyze in the sequel.

First of all, notice that for the radial component there are the two equilibrium solutions $r = 0$, and $r = \sqrt{p}$. The only equilibrium condition for the angular component is $r = 0$, implying that the origin $x = 0$ is the unique equilibrium point.

To analyze the local behavior close to the origin, consider the linear approximation of the dynamics (5.16) transformed into euclidean coordinates. Therefore recall that

$$x_1 = r \cos(\theta), \quad x_2 = r \sin(\theta)$$

and consequently for x_1

$$\begin{aligned} \dot{x}_1 &= \dot{r} \cos(\theta) - r \sin(\theta) \dot{\theta} \\ &= (pr - r^3) \cos(\theta) - r \sin(\theta) (\omega + br^2) \\ &= pr \cos(\theta) - \omega r \sin(\theta) + \mathcal{O}^2(r, \theta) \\ &\approx px_1 - \omega x_2 \end{aligned}$$

and for x_2

$$\begin{aligned} \dot{x}_2 &= \dot{r} \sin(\theta) + r \cos(\theta) \dot{\theta} \\ &= (pr - r^3) \sin(\theta) + r \cos(\theta) (\omega + br^2) \\ &= pr \sin(\theta) + \omega r \cos(\theta) + \mathcal{O}^2(r, \theta) \\ &\approx \omega x_1 + px_2 \end{aligned}$$

Thus, the local dynamics are described by

$$\begin{aligned} \dot{x}_1 &= px_1 - \omega x_2 \\ \dot{x}_2 &= \omega x_1 + px_2 \end{aligned} \tag{5.17}$$

with the Jacobian

$$J = \begin{bmatrix} p & -\omega \\ \omega & p \end{bmatrix} \tag{5.18}$$

and eigenvalues

$$\lambda_{1,2} = p \pm i\omega. \tag{5.19}$$

Hence, locally, for $p < 0$ the solutions of (5.16) are stable spirals, decaying monotonically into the origin, and for $p > 0$ they are unstable spirals, growing up exponentially fast (with characteristic time $t_c = p^{-1}$). For larger values of $r = \sqrt{x_1^2 + x_2^2}$ the nonlinear terms become dominant pushing the growth down, and establishing a limit amplitude with $r = \sqrt{p}$. As the angular component does increase continuously (modulus 2π), the behavior observed in Figure 5.16 is obtained.

Thus, a stable limit cycle is born *out of the blue sky* when the parameter p passes through the critical value $p = 0$. This behavior is illustrated in Figure 5.17.

Some words are in order on the nonlinear dependences on the radial component in (5.16). For the radial dynamics ($\dot{r}$) a pitchfork bifurcation occurs at $p = 0$, with neglect of the negative branch. The associated square-root dependency of the amplitude of the limit cycle is typical for the Hopf bifurcation. Figure 5.18 shows the associated bifurcation (or solution) diagram for the radial component. Filled circles denote the amplitude of the asymptotically stable limit cycle.

With respect to the quadratic dependence of the angular dynamics ($\dot{\theta}$) on the radius, it should be mentioned that it can be derived rigorously using center-manifold and normal form arguments (see e.g. [Sas99; PB95; Per01] and others). For $b = 0$, the angular velocity becomes uniform over the radius,

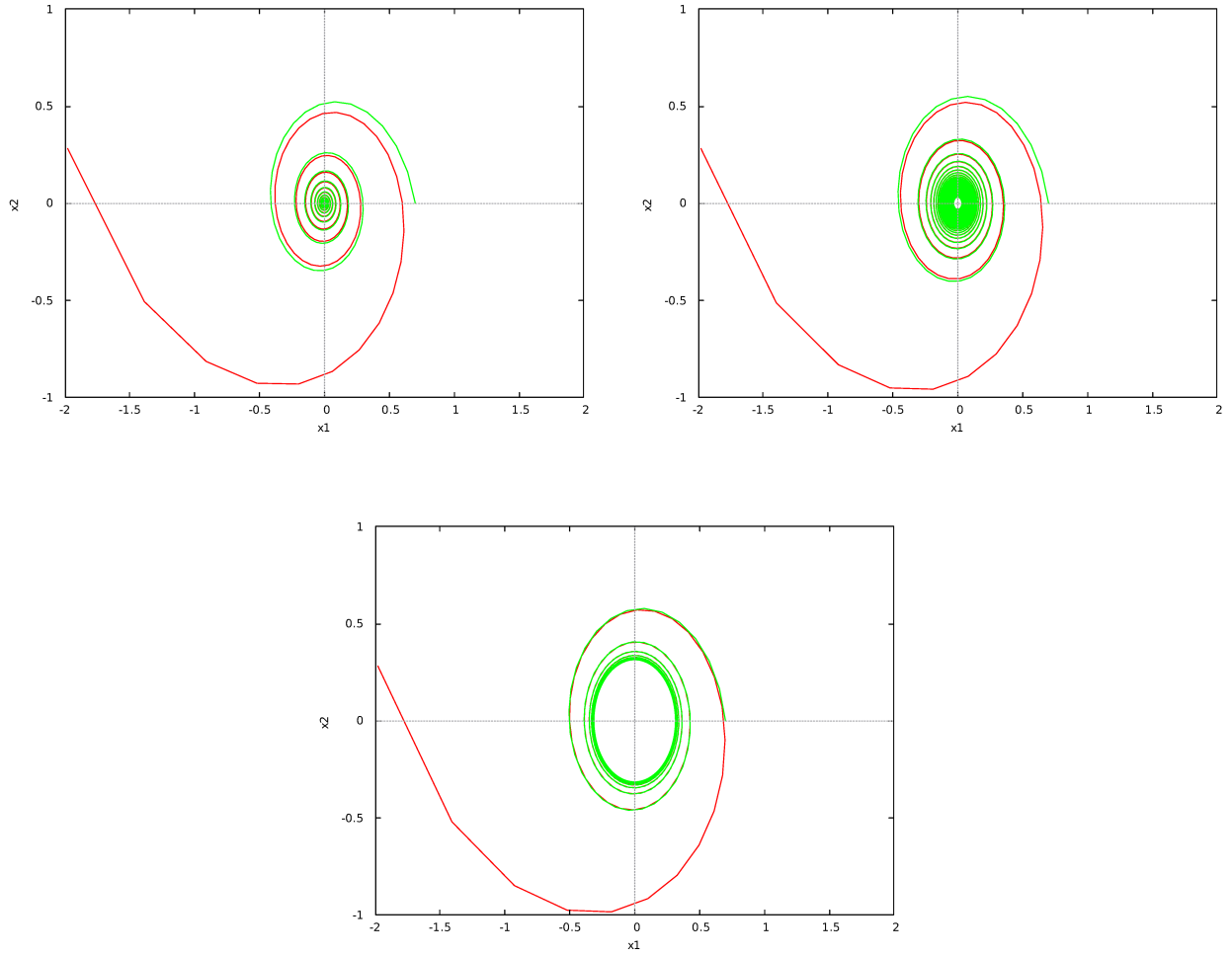


Figure 5.17: Phase plane of the (*supercritical*) Hopf bifurcation associated to the dynamics (5.16) with $p = -0.1$ (top left), $p = 0$ (top right), and $p = 0.1$ (bottom).

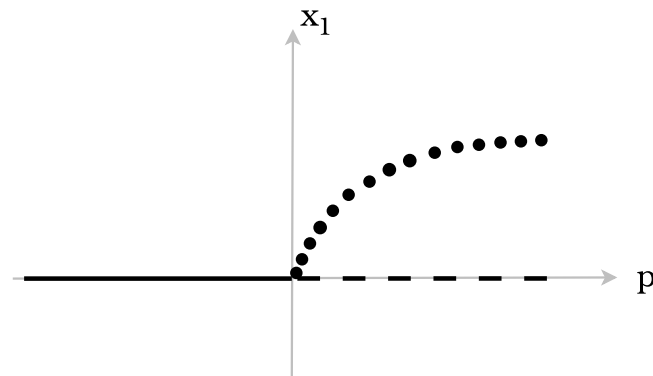


Figure 5.18: Bifurcation diagram of the supercritical Hopf bifurcation associated to (5.16).

and at $r = \sqrt{p}$ the velocity is also uniform, implying that on the limit cycle the dynamics correspond to a uniform oscillator (i.e., (5.10) with $p = 0$).

In the same way as for the standard pitchfork bifurcation there is also a sub- and supercritical Hopf bifurcation. The one discussed above corresponds to the **supercritical Hopf bifurcation**, where the

equilibrium point at the origin runs unstable when the parameter p is moved from negative to positive values, giving rise to an asymptotically stable limit cycle (see Figure 5.17).

The **subcritical Hopf bifurcation** corresponds to the case where the stable origin is surrounded by an unstable limit cycle which collapses at $p = 0$, enclosing the origin and thus leading to a repulsive spiral, while no *stable* limit cycle solution exists around it any more. This situation implies that the trajectories will diverge far away from the origin, once passing the parameter through its critical point. A typical example of this behavior is given by the dynamics

$$\begin{aligned}\dot{r} &= pr + r^3 \\ \dot{\theta} &= \omega + br^2\end{aligned}\tag{5.20}$$

The only difference with (5.16) is that in (5.20) the cubic term in the radial dynamics enters with positive sign, leading to a solution $r = \sqrt{-p}$ which only exists for $p < 0$, but corresponds to an unstable limit cycle (note that the slope at $r = \sqrt{-p}$ is positive, implying instability). The associated bifurcation diagram is illustrated in Figure 5.19, with blank circles denoting the amplitude of the unstable limit cycle.

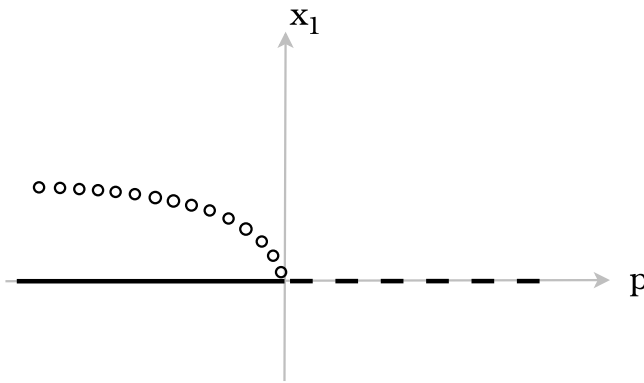


Figure 5.19: Bifurcation diagram of the subcritical Hopf bifurcation associated to (5.20).

A third case may occur, which is called **degenerate Hopf bifurcation**, referring to a conjugate complex eigenvalue passage through the imaginary axis but without the production of isolated periodic orbits (i.e., limit cycles). A typical example is the linear system

$$\begin{aligned}\dot{x}_1 &= px_1 - x_2 \\ \dot{x}_2 &= x_1 + px_2\end{aligned}\tag{5.21}$$

which corresponds to a stable spiral for $p < 0$, a center for $p = 0$ (i.e., a family of periodic orbits), and an unstable spiral for $p > 0$. A nonlinear counterpart of this behavior can be found e.g. for the inverted pendulum with Coulomb friction, taking the friction parameter ρ as bifurcation parameter

$$\ddot{\theta} + \rho\dot{\theta} + k\sin(\theta) = 0.\tag{5.22}$$

The corresponding behavior in phase space is illustrated in Figure 5.20, showing the center-like behavior for $\rho = 0$.

Further interesting phenomena can occur on limit cycles, as discussed in the next section.

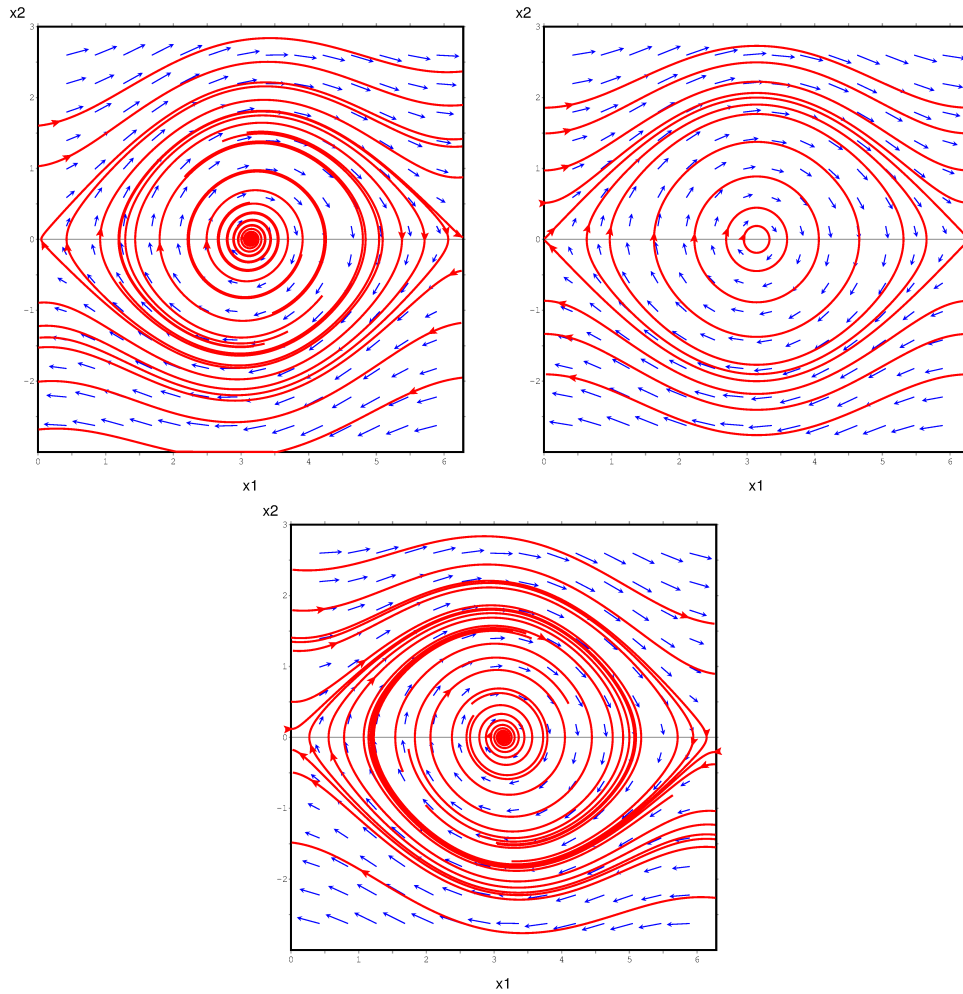


Figure 5.20: Phase plane of the (*degenerate*) Hopf bifurcation associated to the pendulum dynamics with friction (5.22) for $k = 1$: $\rho = -0.1$ (top left), $\rho = 0$ (top right), and $\rho = 0.1$ (bottom).

5.2.3 Bifurcations of limit cycles

In this section the preceding discussion on bifurcations on periodic solutions is extended, including possible bifurcations on the limit cycle, and bifurcations between two or more limit cycles.

5.2.3.1 Infinite period bifurcation

A simple standard dynamics for which a strong bifurcation can be observed on the limit cycle is the following one:

$$\begin{aligned} \dot{r} &= r - r^2 \\ \dot{\theta} &= p - \theta^2 \end{aligned} \tag{5.23}$$

The reader should already recognize that the radial component of the dynamics has two roots, namely $r = 0$ and $r = 1$, with $r = 1$ being the attractive equilibrium. On the other hand, the angular component corresponds to the normal form of the saddle-node bifurcation, like discussed for the non-uniform oscillator in eq. (5.13), indicating that for $p < 0$ there are no equilibrium solutions, and thus the limit cycle with radius $r = 1$ is the unique attractor set. For $p = 0$ something interesting happens on the

limit cycle. In effect, a SS appears at the point $(r, \theta) = (1, 0)$ (or in euklidean coordinates $(x_1, x_2) = (1, 0)$ if θ is defined as illustrated in Figure 5.11), being a (something strange) saddle point with a radial attractive direction (corresponding to the x_1 -axis), and an angular saddle behavior.

For $p > 0$ there are two SSs, one at $(r, \theta) = (1, \sqrt{p})$, and the other at $(r, \theta) = (1, -\sqrt{p})$, with the first one being an attractor, and the second one a repulsor.

Having a look at the origin one observes that for $p < 0$ it is an unstable spiral, while at $p = 0$ the spiraling effect breaks down, and a straight unstable direction appears connecting along the x_1 -axis the unstable origin with the saddle point at $(x_1, x_2) = (1, 0)$. This directions splits of into two when p further increases.

This behavior is qualitatively illustrated in Figure 5.21 for the three cases $p < 0$, $p = 0$, and $p > 0$.

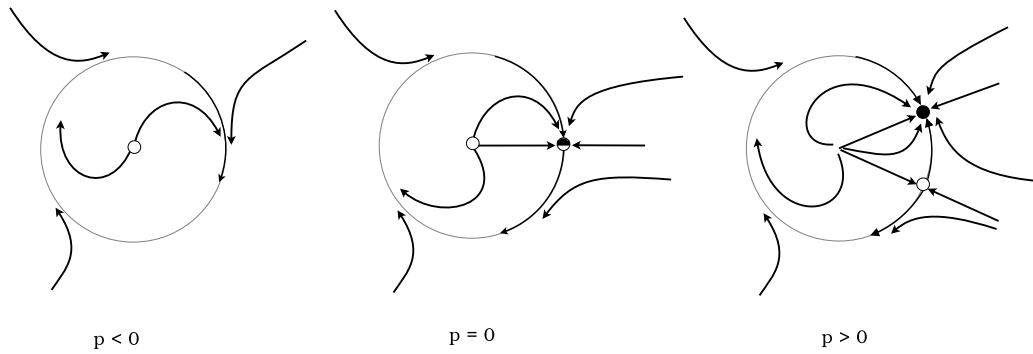


Figure 5.21: Qualitative illustration of the phase plane of (5.23) for different parameter values p with infinite-period bifurcation at $p = 0$.

The associated solution diagram for the pair (r, θ) is shown in Figure 5.22. The filled circles denote the amplitude of the stable limit cycle.

The name for this bifurcation is motivated by the fact, that just before the bifurcation, there appears the bottleneck behavior discussed for the saddle-node bifurcation in (5.13). This leads to an increase in the period which becomes larger when p approaches zero from below (i.e. $p \leq 0$). At the bifurcation point, the time required to move from above of the SS back to the SS once along the circle becomes infinite, given the asymptotic stability behavior. Thus, in this sense, the period becomes infinite, giving rise to the name infinite-period bifurcation.

5.2.3.2 Saddle-node bifurcation of cycles

Another interesting phenomenon which occurs for cycles is the collision between two of them. A typical example is given by the following dynamics

$$\begin{aligned}\dot{r} &= pr + r^3 - r^5 \\ \dot{\theta} &= \omega + br^2\end{aligned}\tag{5.24}$$

For small enough r (i.e., neglecting the fifth order term in the radial component), these dynamics correspond to the subcritical Andronov-Hopf bifurcation normal form (5.16), so that an unstable limit

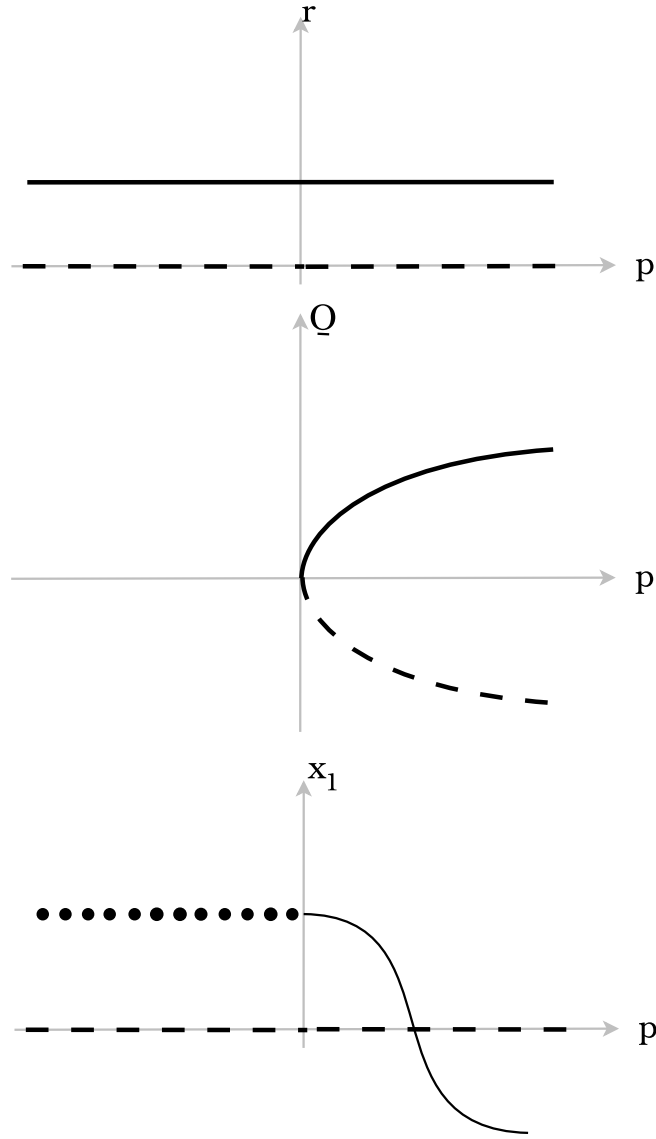


Figure 5.22: Bifurcation diagram for the (r, θ) -components of the infinite-period bifurcation associated to (5.23).

cycle exists for $p \lesssim 0$. Taking into account the complete angular dynamics (i.e., including the fifth order term), this can be rewritten as

$$\dot{r} = r(p + r^2 - r^4)$$

with equilibrium values for r being $r = 0$, or the solution of the quadratic equation in r^2

$$0 = p + \rho - \rho^2, \quad \rho = r^2.$$

The preceding equation has the solutions

$$\rho_{1,2} = \frac{1}{2} \left(1 \pm \sqrt{1 + 4p} \right)$$

showing that for $p < -\frac{1}{4}$ there does not exist any limit cycle and the origin is asymptotically stable. In the range $p \in [-\frac{1}{4}, 0]$, the (asymptotically stable) origin is surrounded by two limit cycles: an unstable

(small) one with radius $r = \sqrt{\frac{1}{2}(1 - \sqrt{1 + 4p})}$, and an asymptotically stable (large) one with radius $r = \sqrt{\frac{1}{2}(1 + \sqrt{1 + 4p})}$. At $p = 0$ the subcritical Hopf bifurcation takes place at the origin, rendering the SS unstable. For $p > 0$ all trajectories converge towards the unique attractor limit cycle with radius $r = \sqrt{\frac{1}{2}(1 + \sqrt{1 + 4p})}$. This behavior is illustrated in Figure 5.23, and the associated bifurcation diagram in Figure 5.24. The filled circles denote the amplitude of a stable limit cycle, while the blank circles the amplitude of an unstable limit cycle.

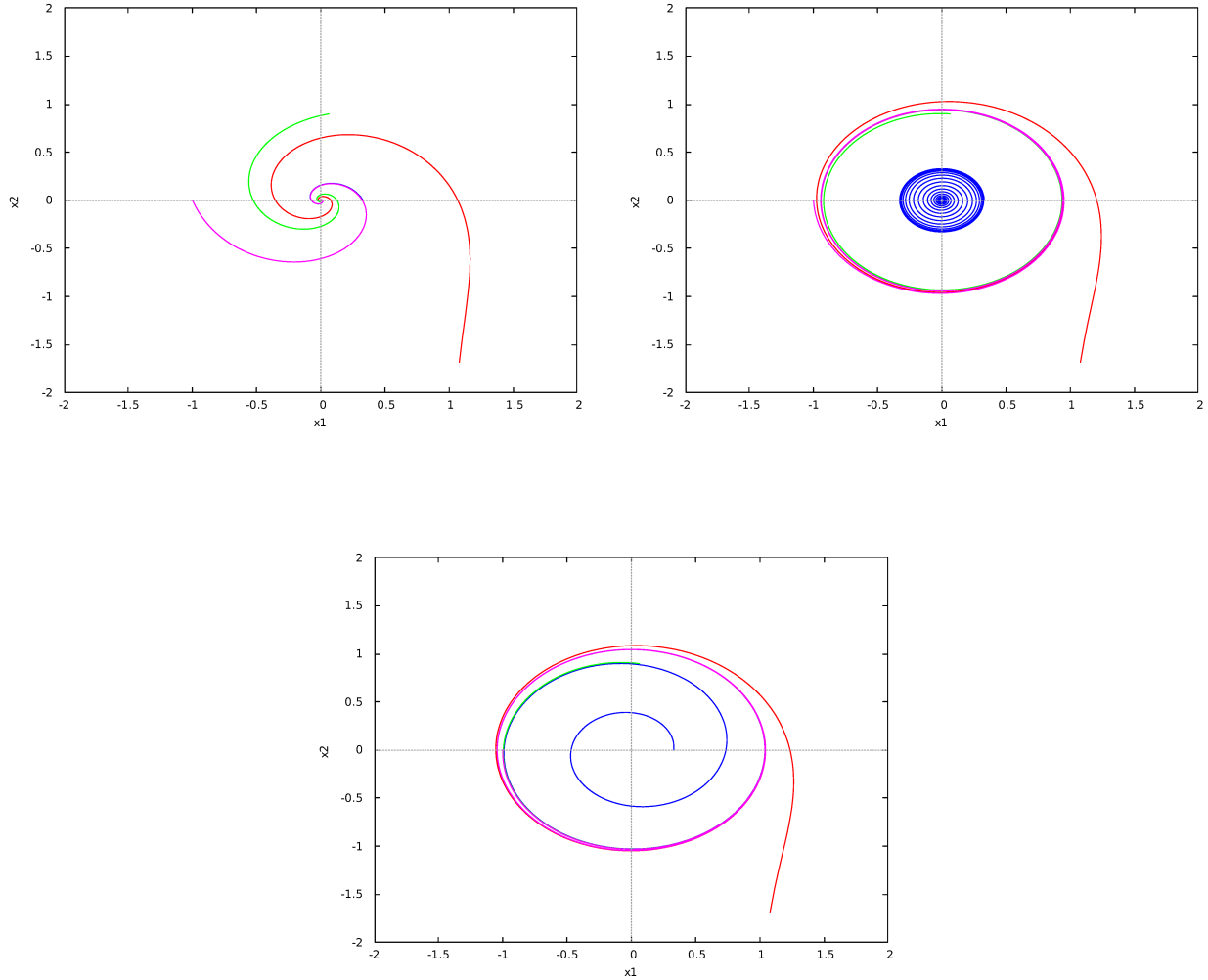


Figure 5.23: Phase plane associated to (5.24) showing a saddle-node bifurcation of limit cycles: $p = -1$ (top left) with stable spiral converging to the origin, $p = -0.1$ (top right) right after the saddle-node bifurcation with a large attractive limit cycle and a repulsor cycle, and $p = 0.1$ (bottom) after the subcritical Hopf bifurcation with large attractive limit cycle.

Note that the bifurcation at $p = -\frac{1}{4}$ is a saddle-node bifurcation of limit cycles, where an unstable and an asymptotically stable limit cycle coincide, leading to the disappearance of both (for $p < -\frac{1}{4}$).

It should be noticed, that the saddle-node bifurcation of limit cycles again goes along with the appearance of a bottleneck, now in the radial component only. This means that just before the bifurcation value for $p \lesssim -\frac{1}{4}$ the trajectories will spiral down to the origin, but the convergence will become very slow about the region where the saddle limit cycle would appear for $p = -\frac{1}{4}$. Hence, if

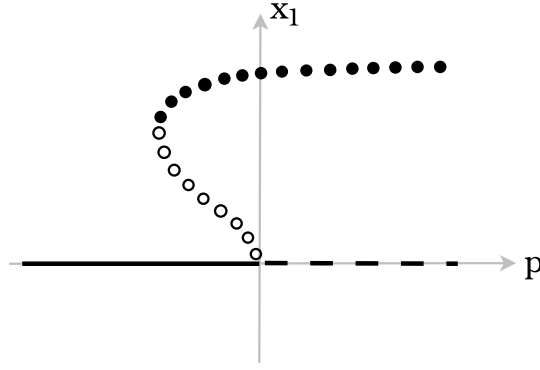


Figure 5.24: Qualitative bifurcation diagram for the radial-component of the saddle-node bifurcation of limit cycles associated to (5.24).

for a given system such a bottleneck behavior is observed, the dynamics should be analyzed for the possibility of saddle-node bifurcations. If the dynamics does not bear the existence of such a behavior, it should be revised in order to use less conservative nonlinear approximations which correspond with the observed behavior.

5.2.3.3 Homoclinic bifurcation

A stable limit cycle, stemming from a supercritical Hopf bifurcation will normally grow with $\sqrt{p}$ as discussed above. In this case the limit cycle can reach a critical amplitude at which it coincides with a neighboring SS. A typical dynamics which leads to such a phenomenon is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + px_2 - x_1^2 + x_1x_2 \end{aligned} \quad (5.25)$$

This system has exactly two SSs for any parameter value p , namely

$$(0,0)' \quad \text{and} \quad (0,1)'.$$

The Taylor linearization around the origin has eigenvalues

$$\lambda_{0,12} = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + 1}$$

and thus is a saddle for any value of $p \in \mathbb{R}$. The linearization about the second equilibrium has the Jacobian, and the associated eigenvalues

$$J = \begin{bmatrix} 0 & 1 \\ -1 & p+1 \end{bmatrix}, \quad \lambda_{1,12} = \frac{p+1}{2} \pm \sqrt{\frac{(p+1)^2}{4} - 1}$$

so that for $p \in (-3, 1)$ the eigenvalues are complex with negative real part for $p \in (-3, -1)$, zero crossing at $p = -1$, and positive real part for $p \in (-1, 1)$. Accordingly, an Andronov-Hopf bifurcation occurs at $p = -1$, given rise to the birth of a limit cycle encircling the equilibrium point $(1,0)'$.

To get a glance of the big picture, one has to figure out what happens to the trajectories leaving the saddle equilibrium at the origin and entering the basin of attraction of the attractive spiral at $(1,0)'$ when $p \in (-3, -1)$. Reasonably they will spiral down to the attractive equilibrium point. Topologically

speaking, when the Hopf bifurcation occurs at $p = -1$, these trajectories will remain in the basin of attraction of the stable limit cycle. This, in turn, increases with increasing p , comming at some value p_* critically close to the saddle point. After a further increase in the bifurcation parameter the limit cycles touches the saddle point, giving rise to a homoclinic trajectory, i.e. a trajectory which connects the saddle with itself. The associated bifurcation is called **homoclinic bifurcation**. A further increase of the parameter p will lead to immediate break-down of the homoclinic trajectory and the disappearance of the limit cycle. This behavior is illustrated in Figure 5.25 using numerical simulation. Note that it is rather difficult to exactly determine the critical parameter value associated to the Homoclinic bifurcation, given that no analytic solution for the limit cycle is at hand.

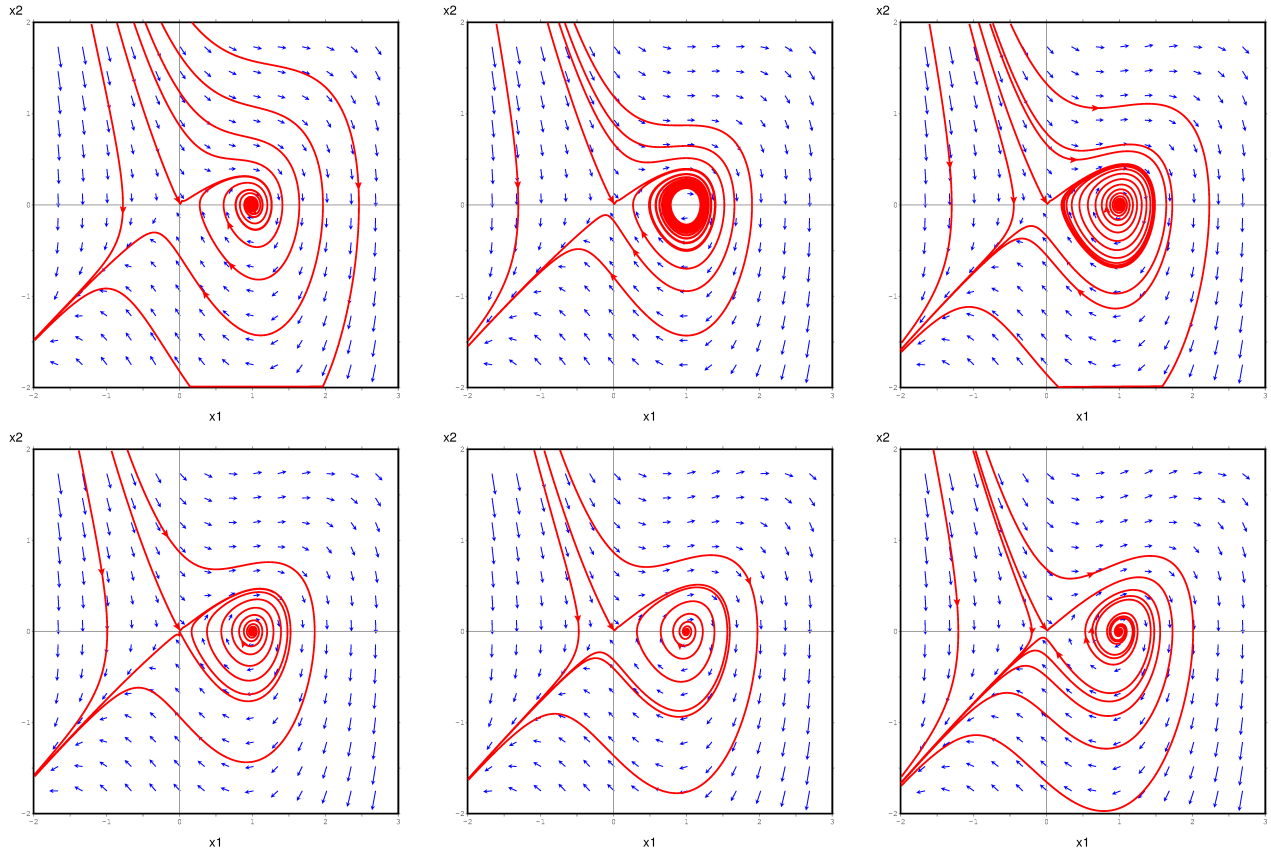


Figure 5.25: Homoclinic bifurcation for the dynamics (5.25): (top left) $p = -1.2$ with stable spiral and saddle node, (top center) $p = -1$ Hopf bifurcation at the point $(1, 0)'$, (top right) $p = -0.9$ the attractive limit cycle approaches the saddle node, (bottom left) $p = -0.8645$ homoclinic bifurcation, (bottom center) $p = -0.8$, and (bottom right) $p = -0.5$ the limit cycles has dissapeared and the trajectories spiral out with time.

5.3 Concluding remarks

There are much more phenomena which are interesting and important to be studied, and the interested reader is referred to more specific literature on this subject like e.g. [Per01], [Wig03] [PB95], [GH77], [Str94] to give just a few ones. In particular the subject of avereging, or weakly nonlinear oscillations is of interest in several applications, and bifurcations associated to more than one parameter which can be simultaneously subject to variations can be very complex, and lead to very strange behavior. Nevertheless, at this point the author hopes that the reader got an idea about the basic bifurcation behavior which nonlinear systems are typically subject to, and stimulated some interest for further reading on this subject.

One particular point should be stressed: Given that in practical applications the model is normally an approximation of the real dynamical behavior, obtained by certain assumptions and simplifications, there might be a gap between the experimentally observed and the predicted behavior. An example is the bottleneck behavior which is not visible in linear systems, and not necessarily in nonlinear ones, if the wrong simplifications have been made. If the analysis or the control of this phenomenon is important for the application, then observation of such a typical behavior should be accompanied by a revision of the modeling assumptions, in order to obtain a more complete model which correctly predicts this behavior. The knowledge of which normal form leads to such phenomena may be an important guide in the particular case.

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Part II

Discrete-time Systems

Basics on discrete-time systems

6.1 Maps, orbits, cobwebs, and fixed points

Systems evolving over time in discrete time steps, like the stock market, computer systems, or systems with discrete time data, like many population balances, and statistical observations, are often modelled using maps, rather than differential equations. In these cases the dynamics of the system are given in the form

$$\mathbf{x}_{t_{n+1}} = \mathbf{f}[\mathbf{x}(t_n), \mathbf{p}], \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (6.1)$$

with $\mathbf{x} : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ being the state of the system, and $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^s$ the map defining how the following state ($\mathbf{x}(t_{n+1})$) evolves from the actual one ($\mathbf{x}(t_n)$).

The sequence of states given by

$$\mathbf{x}(t_0), \mathbf{x}(t_1), \dots, \mathbf{x}(t_n) \quad (6.2)$$

is called the **orbit** of the system, and corresponds to the successive locations in state space where the system states passes through along time. The orbit of a discrete-time system is like the trajectory of a continuous-time one, with the difference that the orbit consists of discrete points, rather than curves. Nevertheless it is often represented through straight lines, connecting the points where the state actually will be found. The orbit of a one-dimensional discrete-time system is illustrated in Figure 6.1

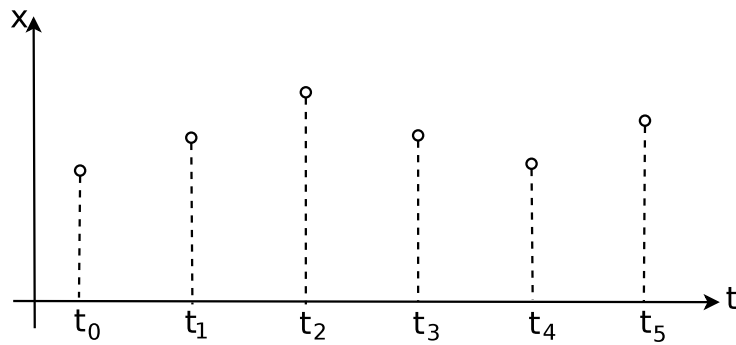


Figure 6.1: Orbit of a one-dimensional discrete-time system.

A very useful way of illustrating and analyzing the dynamics of one-dimensional discrete-time systems is using **cobwebs**, where the subsequent state $x(t_{n+1})$ is plotted against the actual one $x(t_n)$. Figure 6.2 illustrates how to construct a cobweb. One starts at $x(t_0)$, moves to the corresponding point $f(x(t_0), p)$ (1), then to the diagonal line where $x(t_{n+1}) = x(t_n)$ to get $x(t_1)$ (2). From there one moves again to $f(x(t_1))$ (3), and again to the diagonal (4), and so on. This leads to a representation of the orbit in what corresponds to the classical phase space for continuous-time systems.

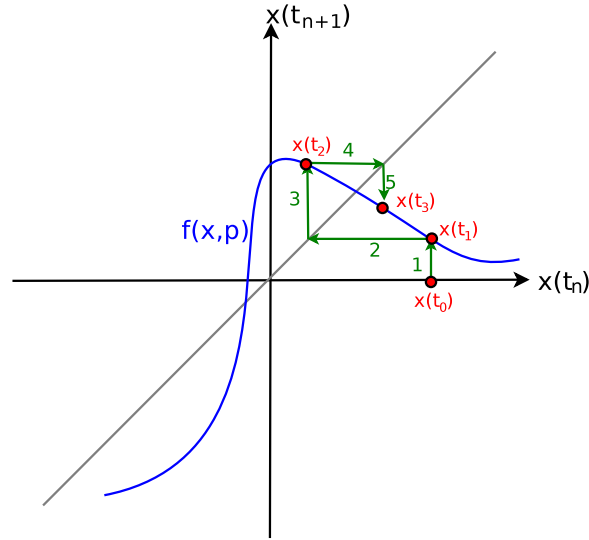


Figure 6.2: Cobweb for a one-dimensional system, illustrating the orbit in the $(x(t_n), x(t_{n+1}))$ -space for the first 4 time steps.

A fundamental difference between continuous- and discrete-time systems consists in the fact that discrete-time systems can present oscillations and very complex, and even chaotic, behavior already in one single dimension. This becomes clear when looking at the dynamics

$$x(t_{n+1}) = -x(t_n), \quad x(t_0) = x_0 \quad (6.3)$$

whose orbit jumps between x_0 and $-x_0$ for all times. In other words, any of the two points on the orbit $x_0, -x_0$ has period 2, i.e. $f(x(t_{n+1})) = x(t_n)$. A particularly interesting phenomenon in discrete-time dynamical systems is the multiplicity of periods which will be discussed later. For the moment we define the period N of an orbit of points $x_1, \dots, x_{N-1}$ as the number of steps after which any point x_i on the orbit is visited again. In the above example 6.3 the period is 2.

A **fixed point** of the dynamical system (6.1) is a point $\mathbf{x}_*$ in state space so that $\mathbf{f}(\mathbf{x}_*) = \mathbf{x}_*$. Clearly, in view of (6.1) this implies that for $\mathbf{x}(t_n) = \mathbf{x}_*$ it holds that

$$\mathbf{x}(t_{n+1}) = \mathbf{x}(t_n) = \mathbf{x}_*$$

and the same is true for any further time, so the orbit is fixed at $\mathbf{x}_*$. Fixed-points are for discrete-time system what equilibrium points (or steady-states) are for continuous-time ones¹. In terms of the period of discrete-time systems, fixed points are period-1 solutions. There may be higher-period solutions, as in the example (6.3), where period-2 solutions are found. The reader should convince himself that period-2 solutions of the map $\mathbf{f}$ are fixed points of the map

$$\mathbf{f}_{[2]} = \mathbf{f}(\mathbf{f}) = \mathbf{f} \circ \mathbf{f} \quad \mathbf{x}(t_{n+2}) = \mathbf{f}(\mathbf{x}(t_{n+1})) = \mathbf{f}_{[2]}(\mathbf{x}(t_n)). \quad (6.4)$$

In the case of (6.3) this is obvious as $\mathbf{x}(t_{n+2}) = \mathbf{x}(t_n)$.

¹Note that in the literature sometimes equilibrium points are also called fixed points, but here we will differ between the names, to distinguish continuous- and discrete-time systems.

In part I of these notes, we already encountered discrete-time systems when we studied periodic linear time-varying systems, and closed orbits with period T . For these, we introduced the notion of the monodromy matrix M , and obtained the first-return (or Poincaré) map (cp. Sections 3.2, and 4.3.2)

$$\mathbf{x}(t+T) = M\mathbf{x}(t), \quad M = \psi(T)e^{TB}P^{-1}(t)$$

which is a linear discrete-time system. The interesting more general connections between Poincaré maps of continuous-time and the associated discrete-time systems can be investigated elsewhere (see e.g. [Str94; Lyn07]), and are definitely worth to be examined.

As has been seen in the first part of these notes, a central point in dynamics is about equilibrium or fixed points. In particular the notion of stability is fundamental, and will be discussed in the next section.

6.2 Stability

The stability notions for discrete-time systems are basically the same as for continuous time system, but formulated in a discrete-time setup. To make this difference clear, the main stability concepts are briefly defined next.

Definition 6.1

A fixed point $\mathbf{x}_*$ is stable, if for any $\epsilon > 0$ there exists a $\delta > 0$, such that for any $\mathbf{x}_0 \in \mathcal{N}_\delta(\mathbf{x}_*)$ it holds that $\mathbf{x}(t_n) \in \mathcal{N}_\epsilon(\mathbf{x}_*)$, i.e.

$$\forall \mathbf{x}_0 : \|\mathbf{x}_0 - \mathbf{x}_*\| \leq \delta \quad \Rightarrow \quad \|\mathbf{x}(t_n) - \mathbf{x}_*\| \leq \epsilon. \quad (6.5)$$

The notion of attractivity is defined as follows

Definition 6.2

A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be attractive in a domain $\mathcal{D}$, if $\forall \mathbf{x}_0 \in \mathcal{D} : \mathbf{x}(t_n) \rightarrow \mathcal{M}$ as $n \rightarrow \infty$.

This definition also covers the case of fixed points $\mathbf{x}_*$ by setting $\mathcal{M} = \{\mathbf{x}_*\}$.

The next definition combines stability and attractivity, leading to the concept of asymptotic stability (which as seen in part I is a central one for application studies).

Definition 6.3

A fixed point $\mathbf{x}_*$ is asymptotically stable if it is stable and attractive.

As discussed in the continuous-time setup (see Section 2.3), asymptotic stability does not provide any convergence measure, and thus it does not allow to establish a time required to approach a desired fixed point sufficiently close for practical purposes. This, in turn, is done by the exponential stability concept. Before defining exponential stability, we should have a look at the analytic solution of an one-dimensional linear discrete-time dynamical system of the form

$$x(t_{n+1}) = \lambda x(t_n), \quad x(t_0) = x_0. \quad (6.6)$$

It is quite easy to see that the solution is given by successive multiplications of λ times the previous state, so

$$x(t_n) = \lambda^n x_0. \quad (6.7)$$

Clearly, in order to achieve stability, it is required that $|\lambda| \leq 1$. For asymptotic stability, λ has to satisfy $|\lambda| < 1$. In this case, it is possible to establish an order of convergence, just as in the exponential stability case for continuous-time systems. Actually, for a 98.5% convergence it is required that

$$\left| \frac{x(t_n)}{x_0} \right| = |\lambda|^n = 0.015$$

what is achieved after

$$n = \log_{|\lambda|}(0.015) = \frac{\ln(0.015)}{\ln(|\lambda|)} \approx \frac{-4}{\ln(|\lambda|)}$$

so that, in comparison with the notion of characteristic and settling times in Section 2.3, the characteristic step number n_c and settling step number n_s can be defined as

$$n_c = \left\lceil \frac{1}{\ln(|\lambda|)} \right\rceil, \quad n_s = 4n_c. \quad (6.8)$$

For example, if $\lambda = 0.5$, $n_c \approx 6$, and actually $\lambda^{n_c} \approx 0.5^6 \approx 0.015$. On the basis of this discussion the concept of exponential stability for discrete-time systems is defined next.

Definition 6.4

A fixed point $\mathbf{x}_*$ is called exponentially stable, if there exist constants $a > 0$ and $\lambda < 1$, so that

$$\|\mathbf{x}(t_n) - \mathbf{x}_*\| \leq a\lambda^n \|\mathbf{x}_0 - \mathbf{x}_*\|. \quad (6.9)$$

As seen above, this concept actually allows to compare the convergence of a nonlinear system with the one of a linear system with well-established convergence time measures.

Finally, the concept of practical stability is put into the discrete-time setup. As in the continuous-time case $\bar{\mathbf{x}}$ will be called the nominal operation fixed point corresponding to the nominal parameter vector $\bar{p}$.

Definition 6.5

The fixed point $\mathbf{x}_*$ is called practically stable, if for given initial deviation and parameter perturbation sizes δ_0 and δ_p , respectively, there exist functions $\alpha \in \mathcal{KL}$ and $\beta \in \mathcal{K}$ so that it holds

$$\|\mathbf{x}_0 - \bar{\mathbf{x}}\| \leq \delta_0, \forall n \geq 0 : \|p(t_n) - \bar{p}\| \leq \delta_p, \Rightarrow \|\mathbf{x}(t_n) - \bar{\mathbf{x}}\| \leq \alpha(\|\mathbf{x}_0 - \bar{\mathbf{x}}\|, t) + \beta(\delta_p). \quad (6.10)$$

In practice, the final deviation size $\beta(\delta_p)$ and maximum transient deviation $\alpha(\|\mathbf{x}_0 - \bar{\mathbf{x}}\|, 0)$ are normally given, so that the question on how large perturbations on initial state and parameter deviations are allowed for is addressed. This may be in particular useful when establishing required experiments, robust system design, etc..

6.3 Linear systems

A first idea on the solution of linear discrete-time system was given in the preceding section (6.6)-(6.7). This will now be extended to the general linear case

$$\mathbf{x}(t_{n+1}) = A\mathbf{x}(t_n), \quad \mathbf{x}(t_0) = \mathbf{x}_0. \quad (6.11)$$

The solution is given by

$$\mathbf{x}(t_n) = A^n \mathbf{x}_0. \quad (6.12)$$

This already makes clear that for discrete-time dynamical systems other convergence behavior is possible, and that for the analysis some different concepts from linear algebra are required. First, note that for a zero matrix $A = 0$ the solution becomes trivial, $\mathbf{x}(t_1) = 0$, independent on the initial condition. This means, that for the zero matrix $A = 0$, in one single time step, the fixed point $\mathbf{x}_* = 0$ is reached. This is a new concept of **convergence in finite time**, which does not appear in linear continuous-time systems. The underlying idea can be exploited for control design in discrete-time systems and is called *dead-beat control*². In the general case, a similar behavior can be observed if the matrix A has zero eigenvalues. In this case, all initial states located in the associated null-space of A will be brought directly (in one time step) to the zero vector.

In contrast, the identity matrix $A = I$ corresponds to the worst case, which still achieves stability, as every state is a fixed point ($A\mathbf{x} = \mathbf{x}$, $\forall \mathbf{x} \in \mathbb{R}^n$). Correspondingly, if A has eigenvalues $\lambda_i = 1$ then the associated eigenspace corresponds to a continuum of fixed points.

After this short introduction, consider the case of n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$ with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ so that $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Choose the eigenvectors $\mathbf{v}_i$ as a basis for the $\mathbb{R}^n$, so that the orthogonality condition $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$ holds, where δ_{ij} is the Kronecker- δ . Then the matrix-state product can be written as

$$\begin{aligned} \mathbf{x}(t_1) &= A\mathbf{x}_0 = \sum_{i=1}^n \lambda_i \langle \mathbf{x}_0, \mathbf{v}_i \rangle \mathbf{v}_i \\ \mathbf{x}(t_2) &= A\mathbf{x}(t_1) = \sum_{i=1}^n \lambda_i \langle \mathbf{x}(t_1), \mathbf{v}_i \rangle \mathbf{v}_i \\ &= \sum_{i=1}^n \lambda_i \left\langle \underbrace{\left(\sum_{j=1}^n \lambda_j \langle \mathbf{x}_0, \mathbf{v}_j \rangle \mathbf{v}_j \right)}_{= \sum_{j=1}^n \lambda_j \langle \mathbf{x}_0, \mathbf{v}_j \rangle \delta_{ij}}, \mathbf{v}_i \right\rangle \mathbf{v}_i \\ &= \sum_{i=1}^n \lambda_i^2 \langle \mathbf{x}_0, \mathbf{v}_i \rangle \mathbf{v}_i \\ &\vdots \\ \mathbf{x}(t_n) &= \sum_{i=1}^n \lambda_i^n \langle \mathbf{x}_0, \mathbf{v}_i \rangle \mathbf{v}_i \end{aligned}$$

In the case of conjugate complex eigenvalues $\lambda = a \pm ib$, the matrix can be transformed so that these correspond to a two-dimensional block of the form

$$A_c = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

²To exemplify this idea, consider the 1D system $x(t_{n+1}) = ax(t_n) + u$. Then the dead-beat controller reads $u(t_n) = -ax(t_n)$, and achieves finite-time convergence in one time step. This explains the illustrative name it receives...

and the corresponding matrix-state product component becomes for any $\mathbf{w} \in \mathbb{R}^2$

$$A_c \mathbf{w} = \begin{bmatrix} aw_1 - bw_2 \\ bw_1 + aw_2 \end{bmatrix}, \quad \|A\mathbf{w}\| = \sqrt{a^2 + b^2} \sqrt{w_1^2 + w_2^2} = |\lambda| \|\mathbf{w}\|.$$

Thus

$$\|A_c^n \mathbf{w}\| = |\lambda|^n \|\mathbf{w}\|,$$

In the case of eigenvalues with multiplicity greater than one, the matrix A can be brought into Jordan normal form by an appropriate transformation. To see what happens in this case, consider a Jordan block of dimension two, i.e.

$$A_j = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad A_j^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$$

Then it holds for any $\mathbf{w} \in \mathbb{R}^2$ that

$$A_j \mathbf{w} = \begin{bmatrix} \lambda w_1 + w_2 \\ \lambda w_2 \end{bmatrix}, \quad \|A\mathbf{w}\| = \sqrt{\lambda^2(w_1^2 + w_2^2) + 2\lambda w_1 w_2 + w_2^2} = |\lambda| \sqrt{(w_1^2 + w_2^2) + 2\frac{w_1 w_2}{\lambda} + \frac{w_2^2}{\lambda^2}}$$

and

$$A_j^n \mathbf{w} = |\lambda|^n \sqrt{(w_1^2 + w_2^2) + 2n\frac{w_1 w_2}{\lambda} + n^2 \frac{w_2^2}{\lambda^2}}.$$

Summarizing, the following general result for linear discrete-time systems is obtained.

Theorem 6.1

The origin $\mathbf{x} = 0$ of the linear discrete-time system (6.11) is stable iff all eigenvalues λ_i , $i = 1, \dots, n$ are contained in the unit circle $\bar{U}_1 \subset \mathbb{C}$, i.e. $|\lambda_i| \leq 1$, $\forall i = 1, \dots, n$. It is asymptotically (and exponentially) stable iff $|\lambda_i| < 1$ (i.e. the eigenvalues are contained in the open unit circle U_1).

In the same way as in the continuous-time case, this result will be of fundamental importance for the local analysis of nonlinear discrete-time systems addressed in the next chapter.

Before starting through to nonlinear systems, a remark on the relation between linear continuous- and discrete-time systems is at place. Consider the linear system

$$\dot{x} = \lambda x, \quad x(0) = x_0, \quad x(t) = e^{\lambda t} x_0$$

Obviously, the solutions satisfy

$$x(t + \Delta t) = e^{\lambda(t + \Delta t)} x_0 = e^{\lambda \Delta t} e^{\lambda t} x_0 = e^{\lambda \Delta t} x(t),$$

and thus an equivalent discrete-time system is obtained by setting $t_n = t$, $t_{n+1} = t + \Delta t$, and reads

$$x(t_{n+1}) = e^{\lambda \Delta t} x(t_n).$$

It should be clear that, if $x(0) = x(t_0)$, the solutions $x(t)$ and $x(t_n)$ are identical in all points $t_n = n\Delta t$, and defining $\gamma := \exp(\lambda \Delta t)$, the stability condition $\lambda < 0$ is equivalent to $\gamma < 1$.

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Nonlinear discrete-time systems

In this chapter, the local behavior of nonlinear discrete-time systems close to fixed points is addressed. For this purpose, the discrete-time versions of the Hartman-Grobman and the Center-manifold theorems are introduced and discussed.

7.1 Hartman-Grobman theorem for maps

Consider the general nonlinear discrete-time system (6.1), and let $\mathbf{x}_*$ denote a fixed point, i.e. $\mathbf{f}(\mathbf{x}_*, p) = \mathbf{x}_*$. Let

$$\mathbf{x}(t_n) = \mathbf{x}_* + \tilde{\mathbf{x}}(t_n), \quad (7.1)$$

be any deviated state close to $\mathbf{x}_*$. Then the dynamics of $\tilde{\mathbf{x}}$ are given by

$$\begin{aligned} \tilde{\mathbf{x}}(t_{n+1}) &= \mathbf{x}(t_{n+1}) - \mathbf{x}_* = \mathbf{f}[\mathbf{x}(t_n), p] - \mathbf{x}_* = \underbrace{\mathbf{f}(\mathbf{x}_*, p)}_{=\mathbf{x}_*} + \frac{\partial \mathbf{f}(\mathbf{x}_*, p)}{\partial \mathbf{x}} \tilde{\mathbf{x}}(t_n) + \mathcal{O}^2(\tilde{\mathbf{x}}(t_n)) - \mathbf{x}_* \\ &= \frac{\partial \mathbf{f}(\mathbf{x}_*, p)}{\partial \mathbf{x}} \tilde{\mathbf{x}}(t_n) + \mathcal{O}^2(\tilde{\mathbf{x}}(t_n)) \end{aligned}$$

or in terms of the Jacobian matrix $J[\mathbf{x}_*, p] = \frac{\partial \mathbf{f}(\mathbf{x}_*, p)}{\partial \mathbf{x}}$

$$\tilde{\mathbf{x}}(t_{n+1}) = J[\mathbf{x}_*, p] \tilde{\mathbf{x}}(t_n) + \mathcal{O}^2(\tilde{\mathbf{x}}(t_n)), \quad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0 = \mathbf{x}_0 - \mathbf{x}_*.$$

Neglecting nonlinear terms, the (Taylor) linearization (or linear approximation) is given by

$$\tilde{\mathbf{x}}(t_{n+1}) = J[\mathbf{x}_*, p] \tilde{\mathbf{x}}(t_n), \quad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0 \quad (7.2)$$

In correspondence to the continuous-time case, if there are any eigenvalues with norm equal to one, then the linear approximation can not be used to predict the local behavior. A simple example is

$$x(t_{n+1}) = x(t_n) - x(t_n)^3, \quad x(t_0) = x_0 \quad (7.3)$$

for which the corresponding cobweb is depicted in Figure 7.1, showing the local asymptotic stability of the origin, while the linearization predicts a continuum of fixed points (eigenvalue $\lambda = 1$). Actually, by looking on a global cobweb, one easily verifies the global asymptotic stability of the origin.

Accordingly, a hyperbolic fixed point is defined as a fixed point $\mathbf{x}_*$ with associated eigenvalues of the Jacobian contained in the unit circle $U_1 \subset \mathbb{C}$. Next, the Hartman-Grobman theorem for maps is stated (see e.g. [Sas99; Per01]).

Theorem 7.1

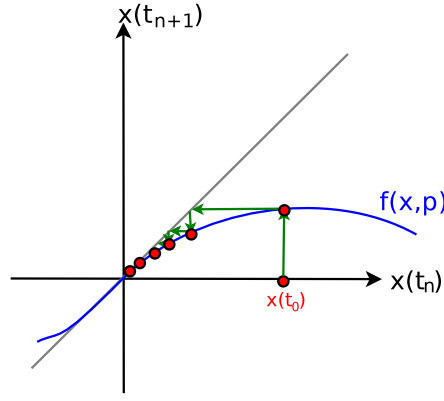


Figure 7.1: Cobweb for the local dynamics of the system (7.3), showing the asymptotic stability property of the origin.

Let $\mathbf{x}_* = 0$ be a hyperbolic fixed point of the map $\mathbf{f}$ (6.1). Then there exists an homeomorphism $\mathbf{h}$ defined on an ϵ -neighborhood $\mathcal{N}_\epsilon(\mathbf{x}_*)$ of $\mathbf{x}_*$, such that $\mathbf{h}[\mathbf{f}(\mathbf{z})] = J[\mathbf{x}_*, p]\mathbf{h}(\mathbf{z})$ for all $\mathbf{z} \in \mathcal{N}_\epsilon(\mathbf{x}_*)$, i.e. the orbits of (6.1) and its Taylor linearization (7.2) are locally topologically equivalent.

Note that in the formulation of the above theorem for simplicity, the fixed point was translated to the origin.

A simple example should illustrate the statement of the Hartman-Grobman theorem 7.1 for maps. Consider the system

$$x(t_{n+1}) = \cos(x(t_n)). \quad (7.4)$$

The fixed point is given by $x_* \approx 0.739$, and the Taylor linearization (7.2) is given by

$$\tilde{x}(t_{n+1}) = -\sin(0.739)\tilde{x}(t_n) \approx -0.67\tilde{x}(t_n).$$

Accordingly, the exact solution $x(t_n)$ of (7.4) and the approximation $x_a(t_n) = x_* + (-0.67)^n(x_0 - x_*)$ should yield similar (or more precisely, topologically equivalent) predictions. This is illustrated in Figure (7.2). In correspondence with Theorem 7.1 it can be appreciated in Figure 7.2 that the solution based on the Taylor linearization yields a good approximation and the same qualitative behavior as the exact one.

7.2 Center-Manifold theorem for maps

As discussed previously, in many cases of interest there are eigenvalues on the unit circle, leading to non-hyperbolic equilibrium points. These cases can not be analyzed using the Hartman-Grobman theorem, and an alternative analysis has to be performed. A central tool for such tasks is the center manifold theorem for maps, which will be discussed next. Note that beyond the case of unit-module eigenvalues, the center manifold approach can also be employed for the analysis of systems with widely separated time scales (as infinite-dimensional parabolic systems [GK05]).

Consider the general discrete-time system (6.1) and let the Jacobian have n_s stable eigenvalues (in the unit circle), and $n_c = n - n_s$ eigenvalues on the unit circle. Then by a linear transformation the system can be brought into the form

$$\begin{aligned} \mathbf{x}_s(t_{n+1}) &= A_s \mathbf{x}_s(t_n) + \mathbf{f}_s(\mathbf{x}_s(t_n), \mathbf{x}_c(t_n)) \\ \mathbf{x}_c(t_{n+1}) &= A_c \mathbf{x}_c(t_n) + \mathbf{f}_c(\mathbf{x}_s(t_n), \mathbf{x}_c(t_n)) \end{aligned} \quad (7.5)$$

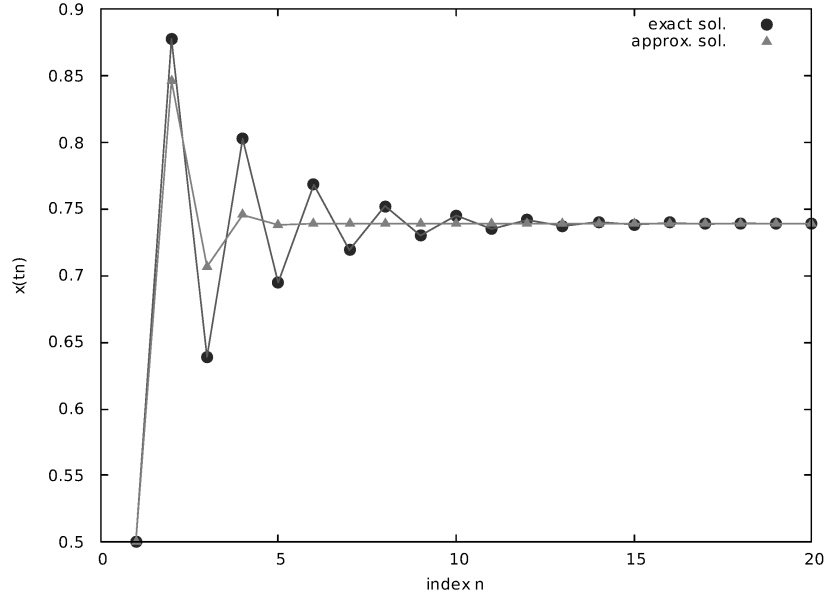


Figure 7.2: Comparison of the local solution of (7.4) using the exact modelling equations (7.4) and the linear Taylor approximation.

with A_s having stable, and A_c having unit module eigenvalues. Given that, locally, the components of the orbit corresponding the directions with stable eigenvalues will converge to a set that contains the origin, implying that after some time, the orbit will depend only on the dynamics on a reduced space, namely the **center manifold**, or in other words, the value of the stable components $\mathbf{x}_s$ will depend on the *center* components $\mathbf{x}_c$, i.e.

$$\mathbf{x}_s = \mathbf{h}(\mathbf{x}_c) \quad (7.6)$$

The center manifold $\mathbf{h}(\mathbf{x}_c)$ has to be tangent to the $\mathbf{x}_c$ -space at the origin, and vanishes there, i.e.

$$\mathbf{h}(0) = 0, \quad \frac{\partial \mathbf{h}(0)}{\partial \mathbf{x}_c} = 0$$

The dynamics of the stable directions on $\mathbf{h}$ can then be written as

$$\begin{aligned} \mathbf{x}_s(t_{n+1}) &= \mathbf{h}(\mathbf{x}_c(t_{n+1})) = \mathbf{h}(A_c \mathbf{x}_c(t_n) + \mathbf{f}_c[\mathbf{h}(\mathbf{x}_c(t_n)), \mathbf{x}_c(t_n)]) \\ &= A_s \mathbf{h}(\mathbf{x}_c(t_n)) + \mathbf{f}_s[\mathbf{h}(\mathbf{x}_c(t_n)), \mathbf{x}_c(t_n)] \end{aligned}$$

Summarizing, the center manifold $\mathbf{h}$ has to satisfy the equation set

$$\begin{aligned} \mathbf{h}(A_c \mathbf{x}_c(t_n) + \mathbf{f}_c[\mathbf{h}(\mathbf{x}_c(t_n)), \mathbf{x}_c(t_n)]) &= A_s \mathbf{h}(\mathbf{x}_c(t_n)) + \mathbf{f}_s[\mathbf{h}(\mathbf{x}_c(t_n)), \mathbf{x}_c(t_n)] \\ \mathbf{h}(0) &= 0, \quad \frac{\partial \mathbf{h}(0)}{\partial \mathbf{x}_c} = 0 \end{aligned} \quad (7.7)$$

The dynamics on the center manifold are given by

$$\mathbf{z}(t_{n+1}) = A_c \mathbf{z}(t_n) + \mathbf{f}_c[\mathbf{h}(\mathbf{z}(t_n)), \mathbf{z}(t_n)], \quad \mathbf{z}(t_0) = \mathbf{z}_0. \quad (7.8)$$

Accordingly, we have the following theorem (cp. [Sas99; Per01]).

Theorem 7.2

Let $\mathbf{f} \in \mathcal{C}^r$, $\mathbf{x}_* = \mathbf{0}$ be a fixed point of (6.1), and its Jacobian at $\mathbf{x}_*$ have s stable eigenvalues, and $z = n - s$ eigenvalues on the unit disk. Then there exist a constant $\delta > 0$, and a function $\mathbf{h} \in \mathcal{C}^r(\mathcal{N}_\delta)$, where $\mathcal{N}_\delta$ is a ball of radius δ around the origin, $\mathbf{h}(0) = \mathbf{0}$, $\frac{\partial \mathbf{h}(0)}{\partial \mathbf{x}} = \mathbf{0}$, that defines the local center manifold

$$\mathcal{W}^c = \{\mathbf{x} = [\mathbf{x}_s, \mathbf{x}_c]^T \in \mathcal{N}_\delta | \mathbf{x}_s = \mathbf{h}(\mathbf{x}_c)\} \quad (7.9)$$

which satisfies (7.7) in $\mathcal{N}_\delta$, and the flow on $\mathcal{W}^c$ is given by (7.8).

The set $\mathcal{W}^c$ is a local attractor set for the orbit of (6.1), and thus, if $\mathbf{x}_* = \mathbf{0}$ is an attractor for (7.8), it is a local attractor for (6.1). Formally, this is reflected in the following theorem (cp. [Sas99]).

Theorem 7.3

If the origin $\mathbf{x}_* = \mathbf{0}$ is asymptotically stable for the dynamics (7.8), then there exist constants $a > 0, \beta < 1$ so that for n large enough it holds that

$$\begin{aligned} \|\mathbf{x}_s(t_n) - \mathbf{h}(\mathbf{z}(t_n))\| &\leq a\beta^n \\ \|\mathbf{x}_c(t_n) - \mathbf{z}(t_n)\| &\leq a\beta^n \end{aligned} \quad (7.10)$$

with $\mathbf{z}(t_n)$ being the solution of (7.8).

These results are illustrated briefly in the following example. Consider the system (cp. (7.3))

$$\begin{aligned} x_1(t_{n+1}) &= \lambda x_1(t_n) + x_2^2(t_n) \\ x_2(t_{n+1}) &= x_2(t_n) - x_2^3(t_n). \end{aligned} \quad 0 < \lambda < 1 \quad (7.11)$$

The system is already in the form (7.5) with $A_s = \lambda$, $A_c = 1$, $f_s = x_2^2$, and $f_c = -x_2^3$. It is clear, that the solution for x_1 will evolve in the following manner

$$\begin{aligned} x_1(t_1) &= \lambda x_1(t_0) + x_2^2(t_0) \\ x_1(t_2) &= \lambda x_1(t_1) + x_2^2(t_1) = \lambda(\lambda x_1(t_0) + x_2^2(t_0)) + x_2^2(t_1) = \lambda^2 x_1(t_0) + \lambda x_2^2(t_0) + x_2^2(t_1) \\ &\vdots \\ x_1(t_n) &= \lambda^n x_1(t_0) + \sum_{i=0}^{n-1} \lambda^i x_2^2(t_{n-1-i}) \end{aligned}$$

and thus, according with Theorem 7.3, after a sufficiently large time (depending on λ) the behavior of x_1 will depend only on x_2 , showing that x_1 converges onto the center manifold, where $x_1 = h(x_2)$.

Similar to the approximation method for continuous-time systems, set a quadratic approximation (in the sense of a Taylor expansion¹)

$$h(x_2) \approx \phi(x_2) = a_2 x_2^2$$

and consider the associated equation (7.7)

$$x_1(t_{n+1}) = h(x_2(t_{n+1})) = a_2(x_2(t_n) - x_2^3(t_n))^2 = a_2 x_2^2(t_n) + \mathcal{O}^4(x_2(t_n)) = \lambda a_2 x_2^2(t_n) + x_2^2(t_n) = (\lambda a_2 + 1)x_2^2(t_n).$$

¹Given the conditions at $x = 0$ in (7.7) the constant and linear terms vanish.

Accordingly a fourth-order approximation is achieved if

$$(\lambda a_2 + 1 - a_2)x_2^2(t_n) = \mathcal{O}^4(x_2(t_n))$$

which holds for

$$a_2 = \frac{1}{1 - \lambda}$$

so that

$$\phi(x_2) = \frac{x_2^2}{1 - \lambda}. \quad (7.12)$$

Recalling the stability analysis of (7.3) the dynamics on ϕ are just the locally asymptotically stable x_2 -dynamics

$$x_2(t_{n+1}) = x_2(t_n) - x_2^3(t_n).$$

Thus, the origin is globally asymptotically stable. It must be noted that in this example the dynamics on the center manifold do not depend on the particular approximation ϕ of h , so that even though the result only predicts the local behavior the stability features apply actually for all initial conditions. Nevertheless, the approximation of the center manifold is a local one, as can be seen in the phase plane plot shown in 7.3. As can be appreciated in Figure 7.3, the quadratic approximation of the center

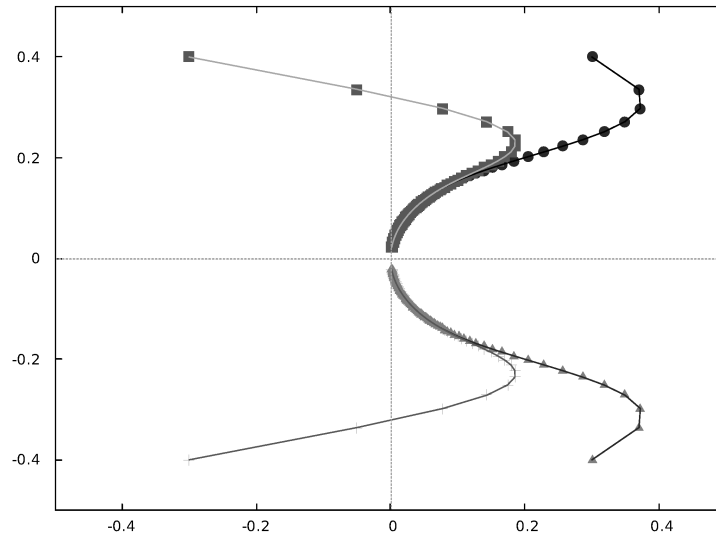


Figure 7.3: Illustration of the orbit in phase space for the two-dimensional nonlinear system (7.11) verifying the local prediction of the center manifold approach.

manifold yields a good approximation of the large time behavior for small initial values. Actually, close to the origin the predicted parabola (7.12) is almost exactly obtained.

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Bifurcations in discrete-time systems

In this section the bifurcation behavior of discrete time systems is analyzed. In the same manner as in continuous-time systems, the basic one-dimensional fixed point bifurcations are studied which are the transcritical, the saddle-node, and the pitchfork bifurcation. Afterwards, the new concept of period doubling bifurcation is introduced. Then, for dimension-two (or higher dimensional) systems, the discrete-time counterpart of the Andronov-Hopf bifurcation, namely the Neimark-Sacker bifurcation is discussed.

8.1 Transcritical bifurcation

The discrete-time transcritical bifurcation occurs in a similar way to its continuous-time counterpart, when two fixed points coincide and interchange stability. The normal form is given by

$$x_{t_{n+1}} = r x(t_n) - x(t_n)^2, \quad x(t_0) = x_0. \quad (8.1)$$

The associated fixed point condition reads

$$0 = (r - 1)x - x^2$$

implying the two solutions

$$x_1 = 0, \quad x_2 = r - 1.$$

According to the Hartman-Grobman theorem 7.1 the fixed point $x_1 = 0$ (or $x_2 = r - 1$) is locally asymptotically stable (or unstable) for $r < 1$, and unstable (or locally asymptotically stable) for $r > 1$. The bifurcation occurs at $r = 1$ where both fixed points coincide in a saddle and the slope of $f(x(t_n), r)$ is equal to one. The solution behavior is best illustrated using a cobweb, as shown in Figure 8.1. The

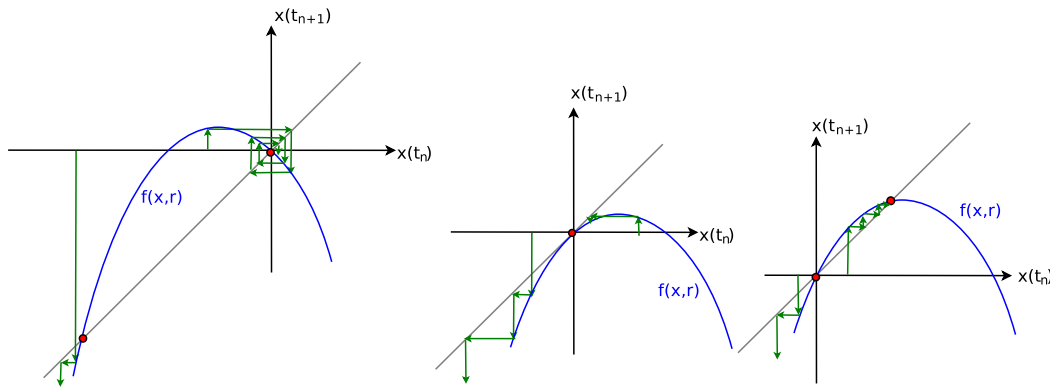


Figure 8.1: Cobweb of the dynamics (8.1) for $r < 1$ (left), $r = 1$ (center, transcritical bifurcation), and $r > 1$ (right).

associated solution diagram is just in correspondence to the one of the continuous-time case, with a parameter shift of 1.

8.2 Saddle-node bifurcation

The normal form of the discrete-time saddle-node bifurcation is given by

$$x(t_{n+1}) = r - x(t_n)^2, \quad x(t_0) = x_0 \quad (8.2)$$

with fixed points

$$x = \frac{1}{2} \left(-1 \pm \sqrt{1 + 4r} \right)$$

where the positive (or negative) fixed point is unstable (or asymptotically stable). The bifurcation occurs at $r = -\frac{1}{4}$ where again the slope of $f(x, r)$ is equal to one at $x = -\frac{1}{2}$. The dynamics of the associated cobweb is qualitatively depicted in Figure 8.2. The associated solution diagram looks just

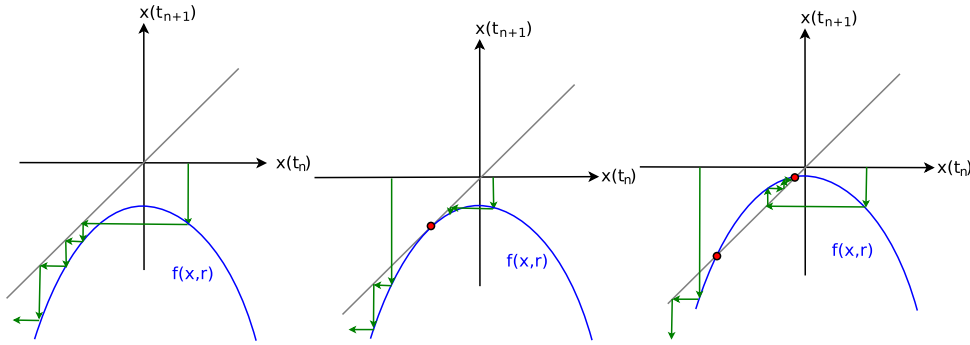


Figure 8.2: Cobweb of the dynamics (8.2) for $r < -\frac{1}{4}$ (left), $r = -\frac{1}{4}$ (center, saddle-node bifurcation), and $r > -\frac{1}{4}$ (right).

the same way as in the continuous time case modulus a parameter shift of $-\frac{1}{4}$.

8.3 Pitchfork bifurcation

The discrete time pitchfork bifurcation has the normal form

$$x(t_{n+1}) = r x(t_n) - x(t_n)^3, \quad x(t_0) = x_0 \quad (8.3)$$

with symmetric fixed point solutions

$$x_1 = 0, \quad x_{2,3} = \pm \sqrt{r-1}.$$

At $r = 1$ the slope of the right-hand side function $f(x, r)$ is equal to one, implying the change of stability of the origin, and the appearance of the other two fixed points. The associated dynamics are qualitatively illustrated using cobwebs in Figure 8.3. The associated solution diagram looks just the same way as in the continuous time case modulus a parameter shift of 1, with $x = 0$ being asymptotically stable for $r < 1$, unstable for $r > 1$, and with two symmetric asymptotically stable solutions $x_{2,3} = \pm \sqrt{r-1}$ for $r > 1$.

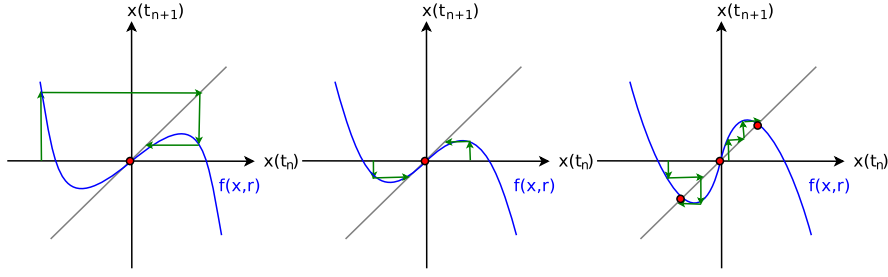


Figure 8.3: Cobweb of the dynamics (8.3) for $r < 1$ (left), $r = 1$ (center, pitchfork bifurcation), and $r > 1$ (right).

In the same way as in the continuous-time case, the above pitchfork bifurcation is called *supercritical*. There is also a *subcritical* pitchfork bifurcation, analogue to the continuous-time case, and its analysis is left to the reader as an exercise.

8.4 Period-doubling bifurcation

The preceding bifurcations occur when the slope of the increment function $f(x, r)$ is equal to $+1$. In the case that the slope is equal to -1 , a different behavior change is observed in the solutions, associated to the oscillation behavior of the orbits. To analyze this, consider the following dynamics

$$x(t_{n+1}) = -rx(t_n) + x(t_n)^3 = f(x(t_n), r), \quad x(t_0) = x_0 \quad (8.4)$$

with Taylor linearization in $x = 0$ given by $-rx$, implying that the slope of the right-hand side is -1 at $r = 1$. The fixed point solutions are given by

$$x_1 = 0, \quad x_{2,3} = \pm\sqrt{r+1}.$$

Clearly, the second pair of fixed points exists only for $r > -1$. It results that a pitchfork bifurcation occurs at $r = -1$ with $x = 0$ being unstable for $r < -1$ (the slope of the right hand side is locally less than -1), asymptotically stable for $-1 < r < 1$, and unstable for $r > 1$ (the slope of $f(x, r)$ is now locally greater than $+1$). The two solution branches are unstable for any $r > -1$. This means that for $r > 1$, there are three unstable fixed points. Thus the orbits can not stay close to fixed points, but somehow jump around inbetween them. Recalling the concept of periodic motions in one-dimensional discrete time systems, it is thus clear that for $r > 1$ no more attractive period 1 solutions, or equivalently fixed points, exist, and we have to look at the period 2 solutions. Therefore, consider the map $f_{[2]}(x, r)$ defined in (6.4), with f defined in (8.4)

$$x(t_{n+2}) = f_{[2]}(x(t_n), r) = f \circ f(x(t_n), r) = r^2 x(t_n) - r(1+r^2)x(t_n)^3 + \mathcal{O}^4(x(t_n)) \quad (8.5)$$

Clearly, period 2 orbits of (8.4) are fixed points of (8.5), and are given by (ignoring fourth-order terms)

$$x_{2,1} = 0, \quad x_{2,23} = \pm\sqrt{\frac{r^2 - 1}{r(1+r^2)}}$$

and clearly exist only for $r > 1$. Comparing (8.5) with the normal form of the supercritical pitchfork bifurcation (8.3), one can see that at $r = 1$ a (supercritical) pitchfork bifurcation takes place, leading to the loss of stability of $x = 0$, and appearance of two attractive period-2 branches. Recalling that fixed points of (8.4) are period-1 orbits, the appearance of period-2 orbits through the bifurcation at $r = +1$ motivates the name *period-doubling bifurcation*, and the normal form of this bifurcation is exactly

(8.4) (see e.g. [Sas99; GH77]). The associated solution diagram for the period-doubling bifurcation is depicted qualitatively in Figure 8.4. The time response of the period doubling dynamics (8.4) is shown

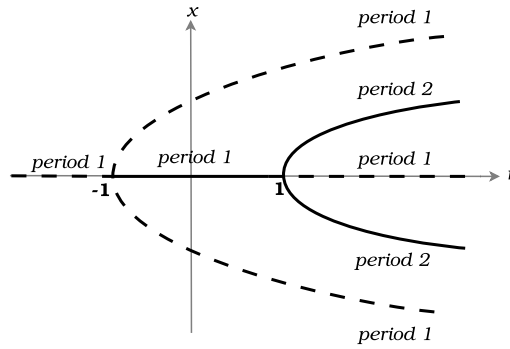


Figure 8.4: Solution diagram for the dynamics (8.4), with period doubling bifurcation at $r = +1$.

in Figure 8.5 for $r = 0$ (before the bifurcation), leading to an attractive period-1 fixed point at $x = 0$, and $r = 1.5$ (after the bifurcation), with two attractive period-2 solutions.

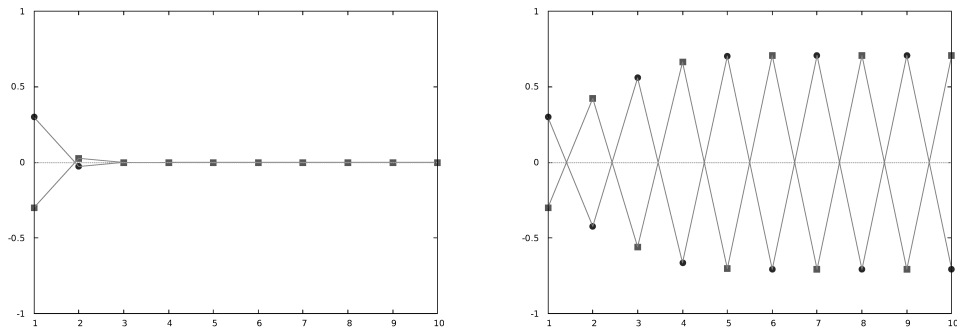


Figure 8.5: Time response of the period doubling dynamics (8.4) with period-1 attractor at $x = 0$ for $r = 0$ (left), and two period-2 attractors for $r = 1.5$ (right).

Period-doubling is an interesting phenomenon, which may e.g. introduce undesired behavior in engineering applications, and plays a key role in the transition to ordered chaos through a set of successive period-doubling bifurcations. An example of this behavior is the logistic map, which will be studied in more detail later.

8.5 Neimark-Sacker bifurcation

In the case of a two- (or more) dimensional dynamics, conjugate complex eigenvalues may appear which can be generically written in the Euler format of complex numbers

$$\lambda_{1,2}(p) = \rho(p)e^{\pm i\gamma(p)}. \quad (8.6)$$

Then, if these eigenvalues pass the unit circle for $p = p_c$ in such a way that $0 < \gamma(p_c) < \pi$, a closed invariant curve appears which is an attractor set for the the orbits of the system. This phenomenon is called *Neimark-Sacker bifurcation*, and is the discrete-time counterpart to the Andronov-Hopf bifurcation. Note that the condition on the angle $\gamma(p_c)$ at the critical parameter value p_c implies that the eigenvalues have to be strictly complex. In this case, defining the complex number

$$z(t_n) = x_1(t_n) + ix_2(t_n)$$

it can be shown, that provided that $\rho'(p_c) \neq 0$ and $e^{ik\gamma(p_c)} \neq 0$ for $k = 1, 2, 3, 4$, the system is locally equivalent to the dynamics

$$z(t_{n+1}) = (1 + \beta)e^{i\gamma(\beta)} z(t_n) + c(\beta)z(t_n)|z(t_n)|^2 + \mathcal{O}^4(|z(t_n)|) \quad (8.7)$$

where β is a new parameter. The situation $e^{ik\gamma(p_c)} = 0$ for any $k \in \{1, 2, 3, 4\}$ is known as strong resonance, and associated to the first four roots of 1 on the complex unit circle. For details on these subtle conditions the reader is referred to more mathematically oriented literature (e.g. [Kuz04; Ioo79]).

Similar to the case of the Andronov-Hopf bifurcation, in the supercritical Neimark-Sacker bifurcation a fixed point loses stability, and a closed orbit appears with increasing radius. There is also the subcritical case, but this will not be treated in the present notes.

To exemplify the Neimark-Sacker bifurcation, consider the dynamics

$$\begin{aligned} x_1(t_{n+1}) &= px_1(t_n) - \frac{1}{2}x_2(t_n) - (x_1^2(t_n) + x_2^2(t_n)) \\ x_2(t_{n+1}) &= \frac{1}{2}x_1(t_n) - px_2(t_n) - (x_1^2(t_n) + x_2^2(t_n)) \end{aligned} \quad (8.8)$$

with Jacobian, eigenvalues and Neimark-Sacker bifurcation threshold

$$J[0] = \begin{pmatrix} p & -\frac{1}{2} \\ \frac{1}{2} & p \end{pmatrix}, \quad \lambda_{1,2} = p \pm i\frac{1}{2}, \quad p_c = \frac{\sqrt{3}}{2}.$$

In Euler notation the eigenvalues are given by

$$\lambda_{1,2} = \sqrt{p^2 + \frac{1}{4}} e^{\pm i \arctan\left(\frac{1}{2p}\right)}$$

and the threshold phase angle is given by

$$\arctan\left(\frac{1}{\sqrt{3}}\right) \approx 0.523$$

implying that the non-resonance condition is met. For $p < p_c$ the origin is a stable spiral, and for $p > p_c$ a closed orbit surrounds the origin with radius which increases with $p - p_c$. This behavior is illustrated in Figure 8.6.

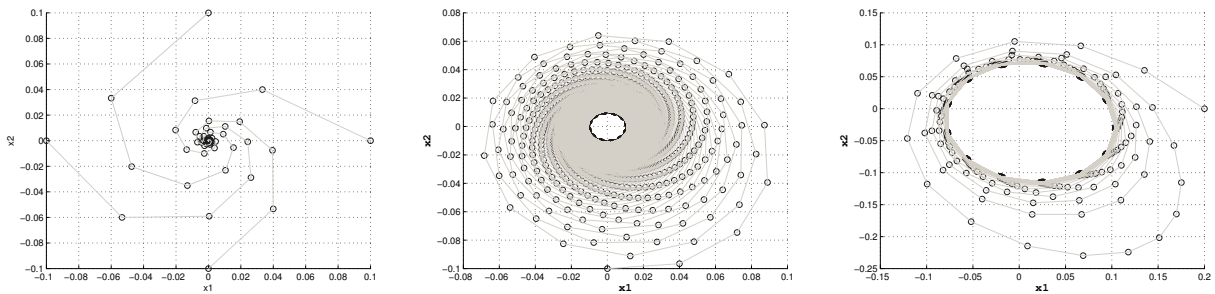


Figure 8.6: Phase plane of the dynamics (8.8) with Neimark-Sacker bifurcation at $p = p_c = \frac{\sqrt{3}}{2}$: (left) $p = \frac{\sqrt{3}}{4} < p_c$, (center) $p = p_c$, and (right) $p = \frac{\sqrt{3}}{2} + 0.01 > p_c$.

It is important to notice that even asymptotically the solution $x(t_n)$ is not necessarily periodic, but converges onto a closed curve γ . On γ , the solution may or not be periodic. The Neimark-Sacker bifurcation does not provide information about periodicity, but about the existence of a closed curve set that attracts the orbit.

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Part III

A brief glance at continuous-time chaos

The Lorenz system

In this part of these notes basic notions of chaos are shortly introduced and discussed starting with the famous Lorenz system [Lor63]. On the basis of these dynamics, some important aspects of chaotic systems are discussed and exemplified.

Edward N. Lorenz (1917-2008) studied cellular convection in the atmosphere [Lor63], ending up with the following simplified model

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}\tag{9.1}$$

where σ is the so-called Prandtl number (i.e. the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity), r the Rayleigh number (i.e. a measure for the ratio of thermal conduction through convection: for large r thermal conduction is primarily through convection, while for low r its due to conduction), and b is related to the aspect ratio of convection rolls.

The first property to be highlighted is the **dissipativeness** of the Lorenz system, meaning that volumes in phase space (or states and trajectories) contract under the flow. To see this, consider the change of an arbitrary volume element $V(t)$ with surface $S(t)$ during the time from t to $t + dt$ subject to the vector field, i.e.

$$V(t + dt) = V(t) + \int_S f(x) \cdot n \, dt \, dA,$$

where dA denotes an infinitesimal area element, and n the outward normal of the surface S at x . By rearranging terms, division through dt , and taking limits, the preceding equation implies

$$\dot{V} = \int_S f(x) \cdot n \, dA = \int_V \nabla \cdot f \, dV,$$

where in the second step the divergence theorem was used. For the Lorenz system (9.1) this becomes

$$\dot{V} = \int_V \left(\frac{\partial}{\partial x} [\sigma(y - x)] + \frac{\partial}{\partial y} [rx - y - xz] + \frac{\partial}{\partial z} [xy - bz] \right) dV = -(\sigma + 1 + b) \int_V dV = -(\sigma + 1 + b) V$$

so that the analytic solution for an arbitrary volume in state space is given by

$$V(t) = V(0) e^{-(\sigma+1+b)t}.$$

Thus, volumes shrink continuously, and exponentially fast. This implies that for any initial conditions trajectories can not move around arbitrarily but are attracted towards some limiting set of vanishing volume (i.e. some planar-like set of thickness zero). What this set may be will be discussed in the sequel. So far we know about equilibrium sets, and periodic closed orbits (limit cycles), but we will see that there are stranger things possible. The volume contraction (dissipativeness) also precludes the

existence of repeller nodes (because they are sources of volume expansion), and thus only attractor nodes, saddles, or closed (planar) periodic motions are possible. Three-dimensional periodic (or quasi-periodic) motions are not possible due to the fact that the dynamics could be viewed as taking place on a torus (cp. [Str94] Ch. 8.6), and the torus would have constant volume in contradiction to the volume contraction, that is independent of the coordinate choice.

The equilibrium points can be explicitly determined. The equilibrium equations read

$$\begin{aligned} 0 &= \sigma(y - x) \\ 0 &= rx - y - xz \\ 0 &= xy - bz \end{aligned}$$

Solving the first and third equilibrium equation, and substituting into the second one, yields

$$0 = (r - 1)x - \frac{1}{b}x^3 = \frac{1}{b}x(b(r - 1) - x^2)$$

which corresponds to the classical normal form of the pitchfork bifurcation (5.3). Accordingly, the origin is an equilibrium point, and there are two additional symmetric equilibrium points at $C^- = (-\sqrt{(r - 1)b}, -\sqrt{(r - 1)b}, r - 1)'$, and $C^+ = (\sqrt{(r - 1)b}, \sqrt{(r - 1)b}, r - 1)'$. The notion C^-, C^+ indicates left- and right-turning convection rolls¹. The associated pitchfork bifurcation takes place at $r = 1$, where the three equilibrium states coalesce. The Jacobian for an arbitrary equilibrium point is given by

$$J[x_*] = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}.$$

Thus, the eigenvalues of the equilibrium point at the origin are given by

$$\lambda_{1,2} = \frac{1}{2} \left(-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4(1 - r)\sigma} \right), \quad \lambda_3 = -b.$$

Thus, for $r < 1$ the origin is a local attractor, and for $r > 1$ a saddle. It is actually possible to show that for $r < 1$ the origin is a global attractor by using the (radially unbounded) Lyapunov function (see Section 4.2.3.5)

$$W(x, y, z) = \frac{1}{2} \left(\frac{1}{\sigma} x^2 + y^2 + z^2 \right) > 0$$

(cp. [Str94]), with change along trajectories given by

$$\begin{aligned} \dot{W} &= -x^2 - y^2 - bz^2 + (r + 1)xy \\ &= -x^2 + (r + 1)xy - \frac{(r + 1)^2}{4}y^4 + \frac{(r + 1)^2}{4}y^4 - y^2 - bz^2 \\ &= -\left(x - \frac{r + 1}{2}y\right)^2 - \left(1 - \frac{(r + 1)^2}{4}\right)y^2 - bz^2 \end{aligned}$$

which for $r < 1$ is non-positive given that it is a negative sum of squares. Actually, the second term vanishes only for $y = 0$ given that $r < 1$, and thus $\dot{W} = 0$ implies $x = 0, y = 0, z = 0$. Hence $\dot{W}$ is negative definite. This, in turn implies that the origin is the unique global attractor for $r < 1$.

¹This notion was originally employed by Lorenz

For the non-zero equilibrium point C^+ the Jacobian becomes

$$J[C^+] = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{(r-1)b} \\ \sqrt{(r-1)b} & \sqrt{(r-1)b} & -b \end{bmatrix}$$

with characteristic equation

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$

Given that no additional equilibrium points can exist, and thus, in case of C^+ (and by symmetry C^-) becoming unstable the next stage of attractor would be a limit cycle (born due to a Hopf bifurcation), we next analyze the possibility of two purely imaginary eigenvalues

$$\lambda_{1,2} = \pm i\omega.$$

Substituting into the characteristic equation yields

$$i(b(r + \sigma)\omega - \omega^3) - \omega^2(b + \sigma + 1) + 2b\sigma(r - 1) = 0$$

so that the critical value of r for a Hopf bifurcation is found to be

$$r_H = \sigma \frac{\sigma + b + 3}{\sigma - b - 1}.$$

Thus, one can conclude that C^+ (and C^-) are local attractors for

$$1 < r < r_H.$$

Next, it is necessary to check if the Hopf bifurcation at $r = r_H$ is subcritical, degenerate, or supercritical. This can be calculated, but requires some additional insight into Hopf bifurcations that go beyond the scope of the present introductory text. Through a set of such calculations (see e.g. [MM76; Dra92]) it can be shown that the bifurcation is *subcritical*. This implies that for $r < r_H$ the equilibrium point C^+ (and C^-) is surrounded by an unstable limit cycle, which collapses over it at $r = r_H$, rendering it unstable (actually it becomes a saddle point). Hence, for $r > r_H$ there are no more stable equilibrium points. The associated solution diagram is depicted in Figure 9.1.

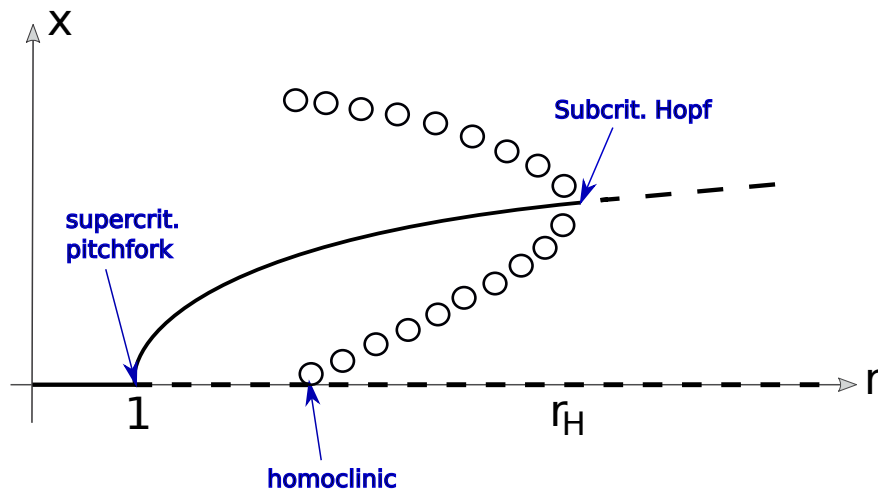


Figure 9.1: Solution diagram for the Lorentz system.

Recall, that the Poincaré-Bendixson theorem does not apply in this case, given that the dimension of the state space is larger than two. This means, that with the means we have studied so far, nothing more can be said on the behavior of trajectories of (9.1) for $r > r_H$. Thus, the next analysis steps are based on numerical simulations.

For $b = \frac{8}{3}$, $\sigma = 10$ and $r = 28 > 24.74 \approx r_H$ the time response shown in Figure 9.2 is obtained, showing **aperiodic long-time behavior**. The question is at place, whether the observed behavior is truly

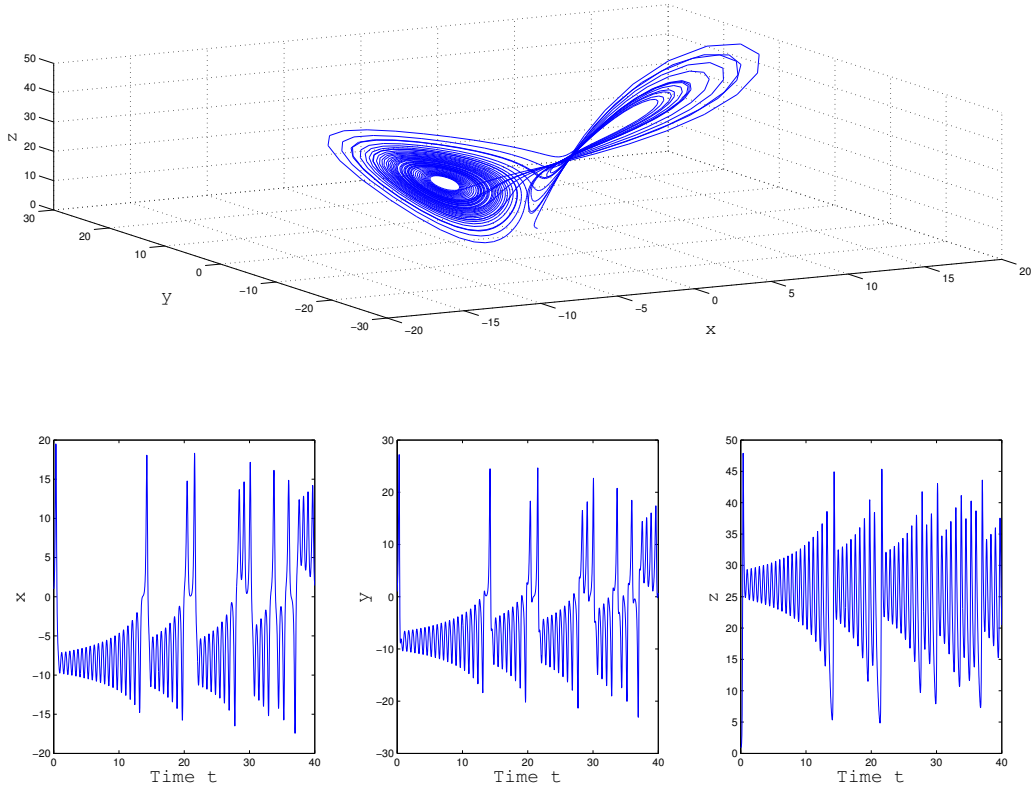


Figure 9.2: Long-time behavior of the Lorenz dynamics (9.1) for $b = \frac{8}{3}$, $\sigma = 10$ and $r = 28 > 24.74 \approx r_H$.

aperiodic or just a long transient towards a periodic motion. To overcome this doubt, Lorenz had the ingenious idea to analyze the discrete-time map defined by the dependency of the maximum of the amplitude of the time response at time t_{n+1} on the maximum in the previous time step t_n , i.e. (see Figure 9.3)

$$z(t_{n+1}) = f(z(t_n)). \quad (9.2)$$

Figure 9.4 shows this, so called Lorenz map taken from a numerical simulation with initial condition $(1, 1, 1)'$, simulation time $T = 10^4$, and parameters $b = \frac{8}{3}$, $\sigma = 10$ and $r = 28 > r_H$. It can be seen that the modulus of the slope $|f'|$ of the map is always greater than 1. Now, taking an arbitrary number p we can see that

$$\begin{aligned} z(t_{n+p}) &= f[z(t_{n+p-1})] \approx f'[z(t_{n+p-1})]z(t_{n+p-1}) = f'[z(t_{n+p-1})]f[z(t_{n+p-2})] \approx \dots \\ &\approx \left(\prod_{i=1}^{p-1} f'(z(t_{n+i})) \right) z(t_n) > z(t_n) \end{aligned}$$

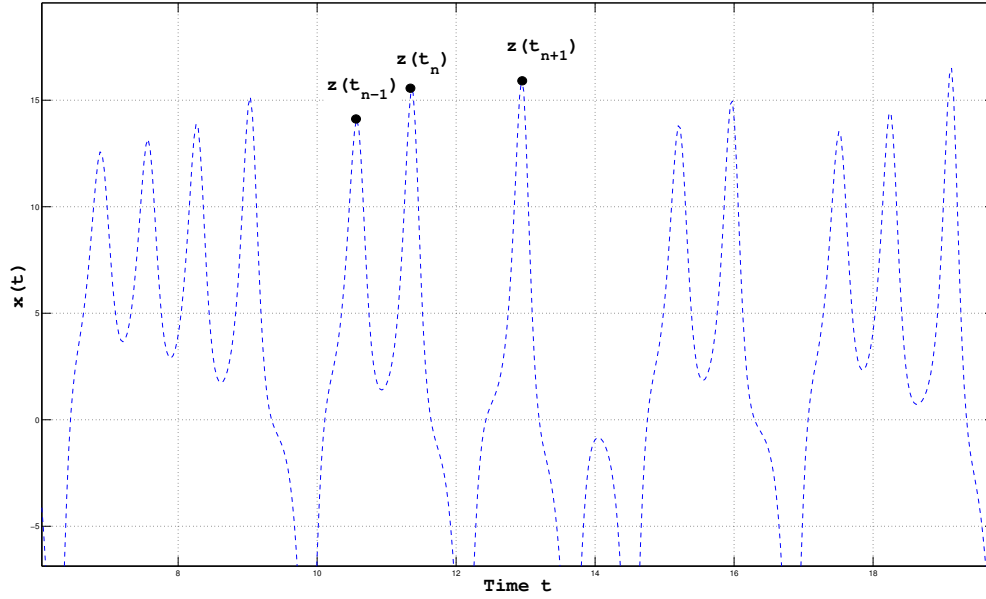


Figure 9.3: Illustration of the meaning of the Lorenz map.

given that $|f'(z)| > 1 \forall z$. Thus, the amplitude changes continuously and no fixed point for the amplitude is approached for any integer $p \in \mathbb{N}$, meaning that any eventual periodic orbit is unstable. Hence, trajectories spiral around the equilibrium points C^+ and C^- , jumping from time to time from one side to the other over and over again (as seen in Figure 9.2), converging meanwhile to an attractor set of zero volume. This attractor set is called a **strange attractor**, to highlight the fact that it is no common attractor set as those we have seen before.

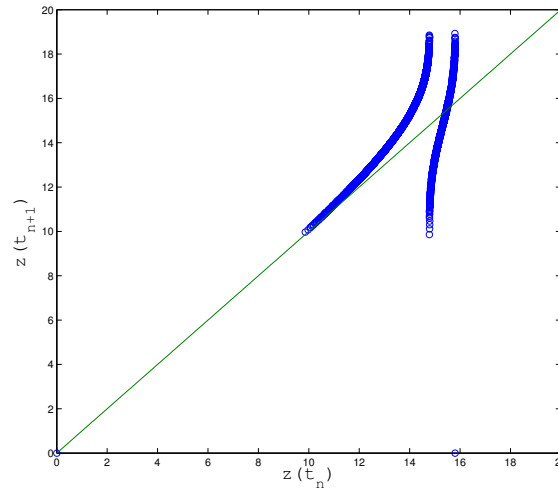


Figure 9.4: Lorenz map (9.2) for $b = \frac{8}{3}$, $\sigma = 10$ and $r = 28 > 24.74 \approx r_H$, initial conditions $(1, 1, 1)'$ and simulation time $T = 10^4$.

It is important to note that the Lorenz dynamics (9.1) are completely **deterministic**, and thus no stochastic influences are responsible for the aperiodic behavior observed in the time response of the system.

Finally, it is interesting to analyze the **sensitivity with respect to initial conditions**, by looking at the distance between trajectories with initial conditions that are close to each other. For this purpose consider the initial conditions

$$\mathbf{x}_1(0) = (10, 10, 10)' \quad \text{and} \quad \mathbf{x}_2(0) = (1.0001, 1.0001, 1.0001)', \quad \mathbf{x} = (x, y, z)'. \quad (9.3)$$

The resulting trajectories and their difference are illustrated in Figure 9.5, showing the natural logarithm of the distance between both trajectories. One observes an exponential growth of the distance

$$\|\delta(t)\| := \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \approx \|\mathbf{x}_1(0) - \mathbf{x}_2(0)\| e^{\lambda t}. \quad (9.4)$$

Actually the growth factor λ in (9.4) for the Lorenz system can be numerically determined as $\lambda \approx 0.9$

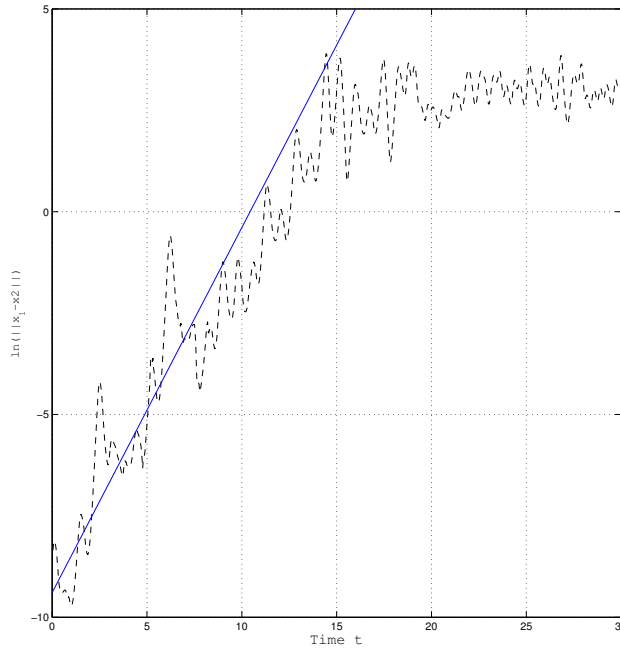


Figure 9.5: Time evolution of the natural logarithm of the difference between the two initial conditions (9.3) for $b = \frac{8}{3}$, $\sigma = 10$ and $r = 28$. The straight line indicates the exact exponential growth with factor 0.9.

(see Figure 9.5). This exponential divergence obviously comes to an end, when the maximal distance is reached that the attractor allows for. This explains that in Figure 9.5 the exponential growth of the difference moves around a fixed value after approximately 15 time units.

To understand the importance of this exponential growth, consider the necessity of predicting the future state of the system by means of only the mathematical model² and note that the time required to leave, say a 10^{-3} tolerance in the prediction for a measurement error in the initial state $\delta_0 = 10^{-7}$ is given by

$$t_{tol} = \frac{1}{\lambda} \ln \left(\frac{10^{-3}}{\delta_0} \right) \approx 10.23.$$

²A better approach to this problem would probably be handled using observers (for an introduction see e.g. [Lue71]).

Now, suppose that that due to a technological investment it would be possible to measure the initial state with about 10^6 times more resolution, so that the initial measurement error would be possible to be set to

$$\delta_n(0) = \delta(0) 10^{-6} = 10^{-13}.$$

Then the prediction time gained by this inversion would be

$$t_{n,tol} = \frac{1}{\lambda} \ln \left(\frac{10^{-3}}{10^{-13}} \right) \approx 25.58$$

so that the factor of improvement is only

$$\frac{t_{n,tol}}{t_{tol}} \approx 2.5,$$

This means that the prediction would be good only 2.5 times longer than the original one. This shows up a great limitation in prediction possibilities for systems with sensitivity to initial errors, like the Lorenz system. Actually, this is one of the standard arguments why good wheather prediction is so hard to achieve.

Summarizing the above analysis of the Lorenz system, the following properties of any chaotic system can be highlighted in a definition-like manner. *A system is chaotic if it shows:*

- (i) **Aperiodic long-term behavior**, i.e. trajectories do not settle down on periodic orbits, equilibrium points, or quasiperiodic orbits (on a torus).
- (ii) **Deterministic dynamics**, i.e. there are no stochastic sources in the system dynamics.
- (iii) **Sensitivity with respect to initial conditions**, i.e. the distance between initial conditions which are close to each other grows exponentially fast.

There are several application examples where very similar dynamics and behavior are obtained, reaching from cooperative catalytic reactions [Pol93], biochemical oscillators (like the Glycolysis) [HM87], brushless DC motors [Hem94], lasers [Hak75], disc dynamos [Kno81], and many more. The reader is encouraged to take a look at the more specialized literature for further insight.

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